THE MATHEMATICAL GAZETTE

EDITED BY
W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc., F.R.S.
AND
PROF. E. T. WHITTAKER, Sc.D., F.R.S.

LONDON
G. BELL & SONS, LTD., PORTUGAL STREET, KINGSWAY, W.C. 2.
AND BOMBAY

Vol. XIV., No. 197. DECEMBER, 1928. 2s. 6d. Net.

CONTENTS.

	PAGE
THE METHODS USED BY ARCHIMEDES FOR APPROXIMATING TO SQUARE ROOTS. T. J. I'A BROMWICH, F.R.S.,	253
NOTE ON THE ASYMPTOTIC EXPANSION OF THE LOGARITHM OF THE FACTORIAL FUNCTION (STIRLING'S SERIES). A. EAGLE, B.Sc.,	258
ELEMENTARY TREATMENT OF LOGARITHMS. T. J. I'A. BROMWICH, F.R.S.,	260
SECOND CONVERSATION ON RIGHT ANGLES. T. C. J. ELLIOTT,	266
MATHEMATICAL NOTES (908-912). T. C. J. ELLIOTT; N. M. GIBBINS, M.A., S. T. SHOVELTON, M.A.; PROF. A. LODGE, M.A.,	268
REVIEWS. NORMAN CAMPBELL, M.A.; H. G. FORDER, B.A.; E. H. LOCKWOOD, M.A.; PROF. T. P. NUNN, D.Sc.; R. S. WILLIAMSON, M.A.	271
GLEANINGS FAR AND NEAR (608-621),	257
THE YORKSHIRE BRANCH,	278
PROPOSED PROBLEM BUREAU,	278
LIBRARY,	279
BOOKS RECEIVED, ETC.,	i

The Mathematical Association.

THE ANNUAL MEETING will be held at the LONDON DAY TRAINING COLLEGE, Southampton Row, London, W.C. 1, on *Monday, 7th January*, 1929, at 3.15 p.m., and *Tuesday, 8th January*, 1929, at 10.0 a.m. and 2.30 p.m.

Intending members are requested to communicate with one of the Secretaries. The subscription to the Association is 15s. per annum, and is due on Jan. 1st. It includes the subscription to "The Mathematical Gazette."

Change of Address should be notified to a Secretary. If Copies of the "Gazette" fail for lack of such notification to reach a member, duplicate copies can be supplied only at the published price.

G. Bell & Sons

Key to Durell's "Elementary Geometry"

Mr. C. V. Durell's "Elementary Geometry" has quickly established itself as one of the most widely-used elementary text-books, and six editions have already been called for. The author has now prepared for the convenience of teachers a book containing solutions of the riders, hints for solution of the numerical exercises, statements of constructions in all cases which are not obvious applications of the bookwork, and those answers which it was considered advisable to omit from the pupil's book.

Now Ready. Price 3s. 6d. net.

Bell's Mathematical Instruments

Messrs. Bell's sets of mathematical instruments have been carefully prepared with a view to putting on the market in each case, at moderate prices, instruments characterised by durability, accurate workmanship, and good finish. Five sets are now ready, ranging in price from 1s. 6d. to 2s. 6d. Messrs. Bell will be pleased to furnish quotations for other sets to meet the special requirements of particular schools.

Five Sets. Prices from 1s. 6d. to 2s. 6d.

Bell's Card of Logarithms and Science Tables

The front gives a table of four-figure logarithms. The back contains (*i*) a table of atomic weights and symbols; (*ii*) a table of sines, cosines, tangents, radians (one degree interval); (*iii*) a number of useful constants and formulæ. The card is intended mainly for class-room use, its size being convenient for holding in the hand, for blackboard work and for handing out to the class, but such that a pupil is unlikely to carry it away.

Size $8\frac{1}{4} \times 9\frac{3}{4}$ *ins. 2s. net per dozen.*

YORK HOUSE, PORTUGAL STREET, LONDON, W.C. 2

THE

MATHEMATICAL GAZETTE.

EDITED BY

W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF

F. S. MACAULAY, M.A., D.Sc., AND PROF. E. T. WHITTAKER, Sc.D., F.R.S.

LONDON:

G. BELL AND SONS, LTD., PORTUGAL STREET, KINGSWAY

AND BOMBAY.

VOL. XIV. DECEMBER, 1928. No. 197.

THE METHODS USED BY ARCHIMEDES FOR APPROXIMATING TO SQUARE-ROOTS.

BY T. J. I'A. BROMWICH, F.R.S.

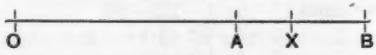
IT is hardly necessary for me to express here my indebtedness to the patience and learning displayed by Sir Thomas Heath in his edition of the *Works of Archimedes* (Cambridge, 1897); but for his statements and suggestions I should never have even known that the greatest master of Greek geometry had left so fascinating a problem for the speculations of later mathematicians. In what follows I can only claim to have tried to frame my conjectures on the hypothesis that so consummate a geometer was at least on the same intellectual plane as such men as Newton or Gauss, so far as a humble admirer can compare those who have towered as giants in the subject.

Taking this view, I submit that Archimedes was at a disadvantage only in that he had a smaller store of accumulated knowledge from which to draw; but, in actual fact, the advance in geometrical knowledge (of the type needed for these solutions) from, say, Euclid to Newton, was extremely slight. The chief advance was in properties of conic sections (which would be of little use in this connexion); and the gain involved in the use of the decimal notation (for calculations of the present kind) is very much less than a modern worker might suppose—in fact, in some recent calculations I have found that decimals are by no means an unmixed blessing.*

1. *First method for finding approximations to* $\sqrt{N}$.

This method is at the bottom geometrical, although (for modern readers) it is probably more easy to state in an algebraic form. Suppose that

$$a < x = \sqrt{N} < b$$



as represented by the diagram where $a=OA$, $x=OX$, and $b=OB$. Then we have

$$0 < (x-a)(b-x) < \tfrac{1}{4}(b-a)^2,$$

and so, if $K=\frac{1}{4}(b-a)^2$, we see that

$$0 < x(a+b) - (N+ab) < K.$$

Thus we find a closer lower limit for x:

$$a_1 = (N+ab)/(a+b).$$

* Of course the decimal notation becomes indispensable for certain modern methods for long computations, which depend on logarithms and calculating machines.

Similarly, we have

$$0<(x-a)^2<K \quad \text{and} \quad 0<(x-b)^2<K,$$

and so in the same way we see that

$$0<(N+a^2)-2ax<K, \quad \text{and} \quad 0<(N+b^2)-2bx<K.$$

Consequently we can use either $(N+a^2)/2a$ or $(N+b^2)/2b$ as the new upper limit b_1; the former is the better when $N<ab$, and the second when $N>ab$.

Archimedes' own problem * corresponds to $N=3$.

It is evident that here $1<x<2$, and so we have

$$a_1=\frac{5}{3}=\frac{3+2\times1}{1+2}<\sqrt{3}.$$

Using this result (which after all might be regarded as obvious) we find an upper limit to $\sqrt{3}$:

$$b_1=\frac{3+(\frac{5}{3})^2}{2(\frac{5}{3})}=\frac{27+25}{30}=\frac{26}{15}.$$

Thus we have now the limits

$$a_1=\frac{5}{3}<\sqrt{3}<\frac{26}{15}=b_1,$$

and then we deduce the closer limits

$$a_2=\frac{265}{153}=\frac{3+\frac{5}{3}\cdot\frac{26}{15}}{\frac{5}{3}+\frac{26}{15}} \quad \text{and} \quad b_2=\frac{3+(\frac{26}{15})^2}{2(\frac{26}{15})}=\frac{1351}{780},$$

using the second form for b_2, because $N=3>a_1b_1=\frac{26}{9}$. Thus

$$\tfrac{265}{153}<\sqrt{3}<\tfrac{1351}{780},$$

which are the limits used by Archimedes.

2. *Second method for the same problem.*

This method, although really quite general, is most easily explained by the special calculations for $\sqrt{3}$; here we have certainly

$$x=\sqrt{3}>1+\tfrac{2}{3}=\tfrac{5}{3}.$$

Write then

$$3x=5+y,$$

so that

$$27=(5+y)^2=25+10y+y^2,$$

giving

$$2=10y+y^2.$$

Thus $y<\frac{1}{5}$; but clearly y is close to this value. Write therefore

$$5y=1-z,$$

and so

$$2=2(1-z)+\tfrac{1}{25}(1-z)^2,$$

or

$$1=52z-z^2.$$

Now $z<1$, so that $z^2<z$, and thus $52z-z<52z-z^2=1<52z$,

from which we see that

$$\tfrac{1}{51}>z>\tfrac{1}{52}.$$

Thus we have two limits:

$$\sqrt{3}>1+\tfrac{1}{3}\{2+\tfrac{1}{5}(1-\tfrac{1}{51})\}$$
$$=1+\tfrac{1}{15}\tfrac{2}{3}=\tfrac{2}{1}\tfrac{6}{5}\tfrac{5}{3}$$

and

$$\sqrt{3}<1+\tfrac{1}{3}\{2+\tfrac{1}{5}(1-\tfrac{1}{52})\}$$
$$=1+\tfrac{5}{7}\tfrac{7}{8}\tfrac{1}{0}=\tfrac{1351}{780}.$$

If the reader cares to follow this process further, he may write $52z=1+w$, which gives $1=2702w-w^2$. Thus w is of the order 1/2700, from which we can prove that the error in using $\frac{1351}{780}$ for $\sqrt{3}$ is less than $\frac{1}{2}\times10^{-6}$ (or 5 in the 7th decimal place).

* The results are used in his estimates for π (*i.e.* $3\frac{10}{71}<\pi<3\frac{1}{7}$); see *Works* (pp. 94 and 96), and the summary in § 4 below.

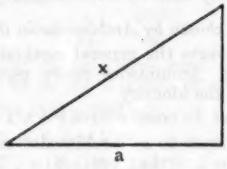
As this second method is more algebraic than the first, it appears rather less likely to be the method actually used by Archimedes. But it certainly could be thrown into a geometrical form more readily than the methods suggested by Zeuthen, Tannery, Heilermann, and others.*

3. *Method for approximating to square roots of large numbers.*

This problem arises in several places when Archimedes is finding limits for π; and these square roots (denoted by x) are all of the type given by

$$x^2 = a^2 + b^2$$

where a, b are large rational numbers and b^2/a^2 is always less than $\frac{1}{12}$.



It is evident that under these conditions $(x-a)$ is small compared with a; and we can approximate by writing our equation in the form

$$x - a = \frac{b^2}{x+a}.$$

From this, by trial,† we can estimate an approximation $x = c_0$; then closer approximations are given by $x = c_1, c_2, c_3, \ldots$ where

$$c_1 - a = \frac{b^2}{c_0 + a}, \quad c_2 - a = \frac{b^2}{c_1 + a}, \quad c_3 - a = \frac{b^2}{c_2 + a}.$$

For then

$$\frac{c_1^2 - a^2}{b^2} = \frac{2a}{c_0 + a} + \frac{b^2}{(c_0 + a)^2}$$

or

$$\frac{c_1^2 - (a^2 + b^2)}{b^2} = \frac{b^2 - (c_0^2 - a^2)}{(c_0 + a)^2} = \frac{(a^2 + b^2) - c_0^2}{(c_0 + a)^2}.$$

Thus the accuracy of the approximation is improved (roughly) by the factor $b^2/(2a)^2$ at each stage, which is an improvement at least of the order 1/50.

To give an example of the method, let us consider the first lower limit obtained by Archimedes, namely, that of ‡

$$\sqrt{(571)^2 + (153)^2}.$$

By trial we might take $c_0 = 571 + 20 = 591$, and then

$$c_1 - a = \frac{(153)^2}{591 + 571} = \frac{23409}{1162} = 20\,\frac{169}{1162}.$$

A little reflexion shows that we get no real improvement by constructing c_2 from c_1, and we can now infer the limit actually used, namely, §

$$\sqrt{(571)^2 + (153)^2} > 591\tfrac{1}{8}.$$

---

* See pp. lxxx-xcix of Sir Thomas Heath's *Introduction* to the *Works*.

† It is usually enough to take $c_0 - a$ as the integer next below $b^2/2a$; thus in the two examples below:

(i) $\frac{b^2}{2a} = \frac{23409}{1142}$ suggesting 20, (ii) $\frac{57600}{3646}$ giving 15.

‡ *Works*, p. 94: these numbers arise from the fact that $\frac{7}{5}\frac{1}{4} = 2 + \frac{2}{7}\frac{6}{5} < 2 + \sqrt{3} = \cot(\pi/12)$.

§ The limit $591\frac{1}{4}$ is *admissible*: but it does not improve the final value, while the subsequent stages would be made slightly heavier by the change from $\frac{1}{8}$ to $\frac{1}{4}$. (See § 4 below.)

As a further example * we take the second upper limit used by Archimedes, which refers to

$$\sqrt{(1823)^2+(240)^2}.$$

By trial we may take $c_0 = 1823+15 = 1838$.

Then $$c_1 - a = \frac{57600}{3661} = 15\frac{2685}{3661}.$$

Since $c_1 > c_0$, we shall clearly have $c_2 < c_1$; now actually $\frac{2685}{3661} < \frac{9}{11}$, so that we can say safely that the required square root is less than †

$$1823+15\tfrac{9}{11} = 1838\tfrac{9}{11},$$

which is the estimate actually used by Archimedes.

4. *Reasons for the fractions chosen by Archimedes in the approximations of* § 3.

It is perhaps easiest to state the general method followed in the modern notation of Trigonometry; Archimedes really uses a geometrical method, which can be expressed by the identity

$$\cot\tfrac{1}{2}\theta = \cot\theta + \operatorname{cosec}\theta = \cot\theta + \sqrt{1+\cot^2\theta}.$$

Then, to find an upper limit to π, Archimedes calculates a lower limit to $\cot(\pi/96)$, starting from the fact that $\cot(\pi/6) = \sqrt{3}$; we have then

$$\cot\left(\frac{\pi}{12}\right) = \sqrt{3}+2 > 2+\frac{265}{153} = \frac{571}{153}.$$

Hence $$\cot\left(\frac{\pi}{24}\right) > \frac{1}{153}\{571+\sqrt{(571)^2+(153)^2}\},$$

and, using the first result of § 3, we see that

$$\cot\left(\frac{\pi}{24}\right) > \frac{1162\frac{1}{8}}{153}.$$

The next stage introduces

$$\sqrt{(1162\tfrac{1}{8})^2+(153)^2} > 1162\tfrac{1}{8}+10,$$

where the method indicated in § 3 is used to evaluate the 10.

Thus $$\cot\left(\frac{\pi}{48}\right) > \frac{1}{153}(1162\tfrac{1}{8}+1172\tfrac{1}{8}) = \frac{2334\frac{1}{4}}{153}.$$

Finally, we prove that $\sqrt{(2334\frac{1}{4})^2+(153)^2} > 2334\frac{1}{4}+5$,

and so $$\frac{96}{\pi} > \cot\left(\frac{\pi}{96}\right) > \frac{1}{153}(2334\tfrac{1}{4}+2339\tfrac{1}{4}) = \frac{4673\frac{1}{2}}{153},$$

which yields $$\pi > \frac{96\times 153}{4673\frac{1}{2}} = 3\frac{667\frac{1}{2}}{4673\frac{1}{2}} < 3\tfrac{1}{7}.$$

It is evident that the effect of the successive steps here is merely to multiply the fraction chosen in the first square root by 4; it will be found, on trial, that to use $\frac{4}{7}$ (in place of $\frac{1}{2}$) in the final form does not enable us to make any closer approximation (of a *simple* type) than $3\frac{1}{7}$. On the other hand, the entire omission of the first fraction makes it impossible to assert that $\pi < 3\frac{1}{7}$. Thus the choice actually made leads to the simplest calculations, consistent with the aim in view.

When dealing with the lower limit for π, upper limits must be found for the square-roots. Thus the first step gives

$$\cot\left(\frac{\pi}{12}\right) < 2+\frac{1351}{780} = \frac{2911}{780},$$

and so $$\cot\left(\frac{\pi}{24}\right) < \frac{1}{780}\{2911+\sqrt{(2911)^2+(780)^2}\}.$$

* *Works*, p. 97.

† The choice of $\frac{9}{11}$ instead of the slightly nearer value $\frac{1}{2}$ is discussed in § 4 below.

The method of § 3 leads without difficulty to the upper limit $3013\frac{3}{4}$ for this square-root; the actual choice of the fraction being probably made so as to allow of a division by 13, because

$$2911+3013\tfrac{3}{4}=5924\tfrac{3}{4}=13\times 455\tfrac{3}{4}.$$

Thus we see that
$$\cot\left(\frac{\pi}{24}\right)<\frac{1823}{240},$$

and so
$$\cot\left(\frac{\pi}{48}\right)<\frac{1}{240}\{1823+\sqrt{(1823)^2+(240)^2}\}.$$

The result of § 3 gives an upper limit $1838\frac{9}{11}$ for this square root; and now the factor 40 divides out, giving

$$\cot\left(\frac{\pi}{48}\right)<\frac{1007}{66}.$$

To explain how this value $\frac{9}{11}$ may have been chosen, suppose that it is replaced by p/q; and then let us try to divide by 40. Now

$$1823+1838=3661=91\times 40+21,$$

and thus we have to adjust p, q, r so that

$$21+\frac{p}{q}=40\times\frac{r}{q},$$

or
$$21q+p=40r.$$

This is the same as
$$p+q=20(2r-q),$$

and the simplest solution will be $p+q=20,\quad 2r=q+1$.

Now from § 3 we see that p/q is not far from $\frac{3}{4}$, and if 20 is divided in this proportion we find the two approximations $p_1=60/7,\ q_1=80/7$; the two nearest *odd* integers to p_1, q_1 are then seen to be $p=9,\ q=11$.

To complete the calculations, we have

$$\cot\left(\frac{\pi}{96}\right)<\frac{1}{66}\{1007+\sqrt{(1007)^2+(66)^2}\},$$

and this square root can be proved * (as in § 3) to be less than $1009\frac{1}{4}$.

Thus
$$\cot\left(\frac{\pi}{96}\right)<\frac{2016\frac{1}{4}}{66},$$

and so *
$$\frac{96}{\pi}<\operatorname{cosec}\left(\frac{\pi}{96}\right)<\frac{1}{66}\sqrt{(2016\tfrac{1}{4})^2+(66)^2}<\frac{2017\frac{1}{4}}{66},$$

or
$$\pi>\frac{96\times 66}{2017\frac{1}{4}}>3\tfrac{1}{7}\tfrac{0}{1}.$$

T. J. I'A. Bromwich.

GLEANINGS FAR AND NEAR.

608. In A^n, the n and A are obvious, and the position of the letters is the symbol of multiplication; but, on the other hand, those who teach the beginner to signify by A^2 the square described on the line A purchase simplicity at the expense of certainty. The same mathematical phrase with them stands for two different things, connected indeed, but of more dangerous consequence from that very connection; for where similarities exist the reader should not be made to convert them into identities. It is of as much importance to impress the distinction of the things signified, as the analogy of their properties. . . . It is bad enough that the word "square" should have both an arithmetical and a geometrical meaning, and causes plenty of confusion: a good notation, if it cannot help in avoiding this confusion, should at least not make it worse.—De Morgan, *Symbols and Notation*, P.C. 23, p. 443.

* After what has been said, the reader will have no difficulty in finding simple reasons for the choice of the two fractions $\frac{1}{4}$ and $\frac{1}{7}$; fuller details are given in pp. lxxxvi-lxxxix of Sir Thomas Heath's *Introduction* to the *Works*.

NOTE ON THE ASYMPTOTIC EXPANSION OF THE LOGARITHM OF THE FACTORIAL FUNCTION (STIRLING'S SERIES).

By A. Eagle, B.Sc.

The following elementary way of obtaining this expansion is interesting in connection with my paper on the Gamma-function in the May issue of the *Mathematical Gazette*.

Assume, according to Stirling's theorem,

$$\Pi(x)=\sqrt{2\pi x}\,(x/e)^x\phi(x)$$

where $\phi(\infty)=1$. Taking logarithms,

$$\log\Pi(x)=\tfrac{1}{2}\log 2\pi+(x+\tfrac{1}{2})\log x \quad -x+\log\phi(x).$$

Changing x into $x-1$ we have

$$\log\Pi(x-1)=\tfrac{1}{2}\log 2\pi+(x-\tfrac{1}{2})\log(x-1) \quad -x+1+\log\phi(x-1).$$

Remembering that $\log\Pi(x)=\log x+\log\Pi(x-1)$, we get, on subtracting the first result from the second,

$$0=(x-\tfrac{1}{2})\log\frac{(x-1)}{x}+1+\log\phi(x-1)-\log\phi(x).$$

Expanding the first logarithm in powers of $1/x$, we get

$$\log\phi(x-1)-\log\phi(x)=\frac{1}{12x^2}+\frac{2}{24x^3}+\frac{3}{40x^4}+\frac{4}{60x^5}+\frac{5}{84x^6}+\dots \quad \dots\dots\dots(1)$$

If we assume, since $\log\phi(\infty)=0$, that

$$\log\phi(x)=\frac{a}{x}+\frac{b}{x^2}+\frac{c}{x^3}+\frac{d}{x^4}+\dots;$$

we get
$$\log\phi(x-1)=\frac{a}{x-1}+\frac{b}{(x-1)^2}+\dots$$
$$=\frac{a}{x}+\frac{a+b}{x^2}+\frac{a+2b+c}{x^3}+\frac{a+3b+3c+d}{x^4}+\dots;$$

and therefore

$$\log\phi(x-1)-\log\phi(x)=\frac{a}{x^2}+\frac{a+2b}{x^3}+\frac{a+3b+3c}{x^4}+\dots.$$

Equating this series to the series in (1), we get $a=1/12$;

$$a+2b=1/12,\quad \therefore\ b=0;$$
$$a+3c=3/40,\quad \therefore\ c=-1/360;$$
$$a+6c+4d=4/60,\quad \therefore\ d=0;$$
$$a+10c+5e=5/84,\quad \therefore\ e=1/1260;\quad \text{and so on.}$$

Whence
$$\log\phi(x)=\frac{1}{12x}-\frac{1}{360x^3}+\frac{1}{1260x^5}-\frac{1}{1680x^7}+\dots.$$

Raising this equation to the exponential, we get

$$\phi(x)=1+\frac{1}{12x}+\frac{1}{288x^2}-\dots.$$

The method adopted above may be generalised as follows: Let u_x be any monotonic decreasing function of x. Then it is obvious geometrically that

$$u_0+u_1+\dots+u_x>\int_0^x u_x dx>u_1+u_2+\dots+u_x.$$

Hence if we denote
$$u_0+u_1+\dots+u_x-\int_0^x u_x dx \text{ by } f(x),$$

we have $f(\infty)$ is finite and equal to C, say. Changing x into $x-1$, we get, on subtracting,

$$u_x - \int_{x-1}^{x} u_x dx = f(x) - f(x-1).$$

If we assume that $f(x)$ can be expanded in the form

$$f(x) = C + \frac{a}{x} + \frac{b}{x^2} + \dots,$$

we get, as in the above example,

$$f(x-1) - f(x) = \frac{a}{x^2} + \frac{a+2b}{x^3} + \frac{a+3b+3c}{x^4} + \dots = \int_{x-1}^{x} u_x dx \quad - u_x,$$

from which a, b, c, ... may be found if the right-hand expression is expanded in powers of $1/x$. Sometimes a series in descending *fractional* powers of x must be assumed for $f(x)$, as is seen, *e.g.* when $u_x = 1/\sqrt{x}$; but the form of

$$\int_{x-1}^{x} u_x dx \quad - u_x$$

will always show the form of expansion to be assumed for $f(x)$.

When several of the coefficients of the series for $f(x)$ have been obtained, we can, by taking a particular value of x, and obtaining both the value of the integral and the sum of the series $u_0 + u_1 + \dots + u_x$ for that value of x, find the approximate value of C.

This method is an elementary way of obtaining the same result as is given by direct substitution in the *Euler-Maclaurin sum formula*.

ALBERT EAGLE.

---

609. The three types of orbit defined by astronomers, namely, the ellipse, the parabola and the hyperbola are, on the face of it, entirely opposed to the laws of gravity.

The comet, even of the short periodic type, approaches the Sun not only along an extended ellipse, but the Sun forms the apex and not the axis of this ellipse. Should the approach of a comet be in the form of a parabola, it is even more pronouncedly the apex, the head stone, it might almost be said, of a pyramid.

To be a consistent satellite of the Sun on the same basis as the planetary system, Halley's comet would require to be seventy-six times the specific gravity of the earth, and over two and a half times that of Saturn, but, as mentioned, this is in any case an impossibility, because it is continuously exhausting its substance and consequently losing centrifugal force.

As to the earth's orbit we have, since civilised man lived on the earth and possessed the calendar, added five days to the year. Formerly the year consisted of 360 days, now it is 365 days odd, and if any one estimating the total weight of our globe takes one-seventy-third of it he will be perhaps able to ascertain the added weight of this enormous series of deposits belonging to the last or Tertiary Stage of the world's existence. (The author is speaking here of meteoric additions to the earth.)—*The Riddle of the Earth*, by "Appian Way," 1925 [per Mr. W. Hope-Jones].

610. For Carnot I feel a great respect. . . . Had I nothing in the wide world but a piece of bread left, I would be proud of sharing it with Carnot.

Carnot invented new tactics, and showed how to fight and conquer with them. While he was engaged in making the giant plans for five armies, he wrote a mathematical work of the highest character [? *Reflexions sur la Métaphysique du Calcul Infinitésimal*, 1797] and composed at the same time some very agreeable little poems. He was a mighty genius indeed!—Lieber, *Reminiscences of G. B. Niebuhr*, 1835, pp. 69, 179.

ELEMENTARY TREATMENT OF LOGARITHMS.

By T. J. I'A. Bromwich, Sc.D., F.R.S.

On previous occasions in the *Gazette* (and in other places) I have repeatedly expressed the opinion that the most natural line of approach to the logarithmic function is the historical one (followed both by Napier and by Newton) in which the logarithm is effectively defined as an integral. From conversation with boys (from eleven to fourteen) I am still convinced that the idea of an *area* is much easier to grapple with than the idea of a *limit*. In spite, then, of other suggestions (which appear in the *Gazette* three or four times a year on the average), I still accept this method as the best: more details will be found in Appendix II. of my book on *Infinite Series* (both editions).

However, I was recently asked by a schoolboy to try to explain to him how the areas could be calculated nearly enough to justify the entries $\log_{10}2 = \cdot30103$, $\log_{10}3 = \cdot47712$, which he encountered in my 5-figure tables (he had been using the ordinary 4-figure tables at school, and was interested to turn over these larger tables). After some thought I showed him the calculations in § 1 below (working only to three places to explain the ideas quickly); we had been led previously to Simpson's rule for finding areas, on the lines indicated at the end of § 1. On testing the method I found it quite easy to extend the calculations to more decimal places (as given here); and, hearing from some friends that the results interested them, I am publishing them in the hope that they may appeal to others.

The use of Simpson's rule for finding areas led me (as set out in § 2) to explain to the same boy how π could be found approximately.

The publication of these two points has led me to add two other notes which have arisen in different ways; the method (given in § 3) for dealing with the logarithmic-sines and logarithmic-tangents of small angles (in my experience with first-year scholars at St. John's College, Cambridge) has almost always proved a useful piece of information to beginners. Finally, there has been of late a recrudescence of the German fable that Napier's methods were substantially anticipated by J. Burgi (the Swiss watchmaker); this involves the statement that Napier used the base $(1-1/q)^q$ with $q=10^7$, while Burgi used $(1+1/p)^p$ with $p=10^4$. The latter statement is true: the former is merely caused by an insufficient study of Napier's actual methods. As far as I know, Napier never published the value of $(1-1/q)^r$ for any higher index r than $r=100$; and he never used the idea of a *base* (in the modern sense) at all. It seems very doubtful whether even Henry Briggs (who published the first table of logarithms to base 10) would have defined $y=\log_{10}x$ by means of the equation $x=10^y$.

For Napier's chief purpose (which was that of calculating spherical triangles) there was little to be gained by using the base 10: thus, if we have to calculate (say) A from a, b, B by using the formula

$$\frac{\sin A}{\sin B}=\frac{\sin a}{\sin b},$$

Napier's logarithms were quite as easy to use as the modern ones (and they had the advantage of *not* requiring *negative characteristics* at all). But (as Briggs himself is careful to state) Napier had already observed that a change in definition would be advantageous for calculating with numbers (instead of sines and cosines); however all this has been set out fully in other places.

1. *Numerical estimates of some logarithms.*

As stated before, I prefer to define the logarithmic function by the integral

$$\log x=\int_1^x \frac{d\xi}{\xi};$$

that is, by means of the area of the rectangular hyperbola $y=1/x$. I have observed recently that this method enables us to obtain some of the more interesting logarithms with very little labour.

Consider, for instance, the logarithm

$$L=\log\frac{9}{8}=2\log 3-3\log 2.$$

Then L is the area of the rectangular hyperbola from $x=8$ to $x=9$; and using, as an approximation, the parabola through the points of the hyperbola which correspond to $x=8$, $8\frac{1}{2}$, 9, we find that

$$L=\frac{1}{6}\left(\frac{1}{8}+\frac{4}{8\frac{1}{2}}+\frac{1}{9}\right).$$

Thus
$$L=\tfrac{1}{6}\,(0{\cdot}125+0{\cdot}4705882\ldots+0{\cdot}1111111\ldots)$$
$$=\tfrac{1}{6}\,(0{\cdot}7066993)=0{\cdot}117783.$$

Similarly, by considering the areas M, N from $x=9$ to $x=10$, and from $x=15$ to $x=16$, we see that

$$M=\log\tfrac{10}{9}=\log 10-2\log 3$$
$$=\frac{1}{6}\left(\frac{1}{9}+\frac{4}{9\frac{1}{2}}+\frac{1}{10}\right)=0{\cdot}105361,$$

and
$$N=\log\tfrac{16}{15}=5\log 2-\log 3-\log 10$$
$$=\frac{1}{6}\left(\frac{1}{15}+\frac{4}{15\frac{1}{2}}+\frac{1}{16}\right)=0{\cdot}065438.$$

The formulae for L, M, N contain the three logarithms $\log 2$, $\log 3$ and $\log 10$; and it seems to be easiest first to eliminate $\log 3$ by forming

$$P=L+M,\quad Q=L+2N,$$

thus
$$P=L+M=\log 10-3\log 2=0{\cdot}223144,$$

and
$$Q=L+2N=7\log 2-2\log 10=0{\cdot}246859.$$

Next we can solve for $\log 2$ by eliminating $\log 10$, and so we have

$$\log 2=2P+Q\quad=0{\cdot}693147,$$

and then
$$\log 10=3\log 2+P=2{\cdot}302585.$$

Hence, by division,* we find that

$$\log_{10}2=\frac{0{\cdot}693147}{2{\cdot}302585}=0{\cdot}30103.$$

We see also from N that

$$\log 3=5\log 2-\log 10-N=1{\cdot}098612,$$

and so
$$\log_{10}3=\frac{1{\cdot}098612}{2{\cdot}302585}=0{\cdot}47712.$$

It appears to be of some interest to show that these familiar logarithms could be obtained from first principles at the expense of such small labour. Of course we cannot estimate the actual errors in L, M, N without using more elaborate methods; but it can be proved that these values are correct to the sixth place of decimals.

The only assumption in the foregoing which may be regarded as difficult is the parabolic formula $S=\frac{1}{3}h(y_1+4y_2+y_3)$ used as an approximation for the area of a rectangular hyperbola.

* It is evident that the logarithmic property, $\log(XY)=\log X+\log Y$, will persist if we divide by any factor μ; and (if $\mu=\log 10$) in the new scheme we shall have unity for $\log 10$.

As a first test of its accuracy the method of square-counting may be applied to any simple graphs which have been drawn previously on squared-

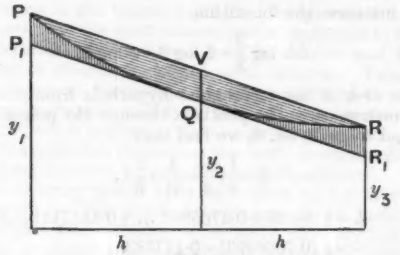


FIG. 1.

paper (such as x^2, x^3, $1/x$). But several boys have found it easy to accept the result as *likely* to be true, when written in the form

$$h(y_1+y_2)-S=2(S-2y_2h).$$

The left-hand side represents the area $PQRV$ (between the curve PQR and its chord PVR), while the area on the right is that between the curve and its tangent P_1QR_1; it is, of course, only in the parabola that the tangent at the mid-point is parallel to the chord.

2. *Two approximations to* π.

The success of Simpson's rule in § 1 led to some attempts to apply the same rule to find the area of a half-quadrant of a circle (and so to deduce an approximation to π); but it was found that the approximation was not very good. The simplest method which I have noticed is the following. We have

$$\sin x=x-\tfrac{1}{6}x^3+\tfrac{1}{120}x^5+O(x^7),$$
$$\tan x=x+\tfrac{1}{3}x^3+\tfrac{2}{15}x^5+O(x^7),$$

so that
$$\tfrac{1}{3}(2\sin x+\tan x)=x\left\{1+\frac{x^4}{20}+O(x^6)\right\}.$$

Further, if $x=\dfrac{\pi}{24}$, the value of $\dfrac{x^4}{20}$ is about $\dfrac{1}{75\times10^3}=\dfrac{1\cdot3}{10^5}$ say.

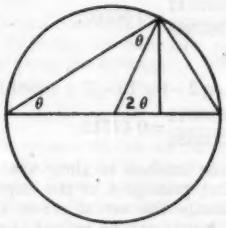


FIG. 2.

Using the formulae* $\sin^2\theta=\frac{1}{2}(1-\cos 2\theta)$, $\cos^2\theta=\frac{1}{2}(1+\cos 2\theta)$, and starting from $\cos(\pi/6)=\cos 30^\circ=\sqrt{3}/2=0{\cdot}8660$, we can easily see that if

$$x=\pi/24=7^\circ\ 30',$$

* Derived at once from the figure, which gives $1+\cos 2\theta=2\cos^2\theta$, $1-\cos 2\theta=2\sin^2\theta$.

then $\sin x = 0{\cdot}13053, \quad \tan x = 0{\cdot}13165.$

Thus $2\sin x + \tan x = 0{\cdot}3927$ nearly,

so that $\pi = 8\,(0{\cdot}3927) = 3{\cdot}1416.$

By drawing the figure of a sector of a circle, its chord and the tangent at the mid-point, the principle already used at the end of § 1, makes the above approximation more or less intuitive, when written in the form

$$2\,(x - \sin x) = \tan x - x.$$

Another method (less easy to suggest graphically) is to use the approximation due to Snellius. For we have

$$\tfrac{1}{8}\,(8\sin x - \sin 2x) = x\,\{1 - \tfrac{1}{30}x^4 + O\,(x^6)\}.$$

Hence with $x = \pi/48$, we find the approximation

$$\pi = 8\left(8\sin\frac{\pi}{48} - \sin\frac{\pi}{24}\right),$$

the relative error being less than $1/10^6$.

By applying the formulae for successive bisection of an angle (using only the arithmetical process of finding square-roots), we obtain

$$\begin{aligned}\pi &= 8\,(0{\cdot}5232250 - 0{\cdot}1305262)\\ &= 8\,(0{\cdot}392688)\\ &= 3{\cdot}1415904.\end{aligned}$$

If one accepts the principle that $\sin x$ is an *odd* function of x (which is not hard to justify), it can be assumed that

$$\sin x = x + cx^3 + O\,(x^5)$$

without knowing c.

Then $$8\sin x - \sin 2x = 6x + O\,(x^5)$$

follows, and so the approximation of Snellius is justified; but this is obviously a more analytical method than the former.

3. *The evaluation of the logarithmic-sine and logarithmic-tangent of small angles.*

Since $$\frac{\sin x}{x} = 1 - \frac{x^2}{6} + \frac{x^4}{120} + O\,(x^6),$$

we have $$\log\left(\frac{\sin x}{x}\right) = -\frac{x^2}{6} + \frac{x^4}{120} - \frac{1}{2}\left(\frac{x^2}{6}\right)^2 + O\,(x^6)$$

$$= -\frac{x^2}{6} - \frac{x^4}{180} + O\,(x^6).$$

Similarly, $$\frac{\tan x}{x} = 1 + \frac{x^2}{3} + \frac{2x^4}{15} + O\,(x^6),$$

so that $$\log\left(\frac{\tan x}{x}\right) = \frac{x^2}{3} + \frac{2x^4}{15} - \frac{1}{2}\left(\frac{x^2}{3}\right)^2 + O\,(x^6)$$

$$= \frac{x^2}{3} + \frac{7x^4}{90} + O\,(x^6).$$

Also $$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + O\,(x^6),$$

and so $$\log\,(\cos x) = -\frac{x^2}{2} + \frac{x^4}{24} - \frac{1}{2}\left(\frac{x^2}{2}\right)^2 + O\,(x^6),$$

$$= -\frac{x^2}{2} - \frac{x^4}{12} + O\,(x^6),$$

or $$\log\,(\sec x) = \frac{x^2}{2} + \frac{x^4}{12} + O\,(x^6).$$

Thus $$\log\left(\frac{\sin x}{x}\right)+\frac{1}{3}\log(\sec x)=\frac{x^4}{45}+O(x^6);$$

and $$\log\left(\frac{\tan x}{x}\right)-\frac{1}{3}\log(\sec x)=\frac{x^4}{45}+O(x^6);$$

thus the last result gives a check on our work, since

$$\log\left(\frac{\sin x}{x}\right)+\log(\sec x)=\log\left(\frac{\tan x}{x}\right) \text{ exactly.}$$

Now, up to about 2°, the value of $\frac{1}{45}x^4$ does not exceed 3 in the eighth decimal place; thus the approximations

$$\log_{10}(\sin x)=\log_{10}x-\tfrac{1}{3}\log_{10}(\sec x) \quad \log_{10}(\tan x)=\log_{10}x+\tfrac{2}{3}\log_{10}(\sec x)$$

have at most an error of about 1·3 ($=3\times\cdot434$) in the eighth decimal place. It is convenient, as a rule, to express x in seconds of arc; and if n'' is the angle, we have

$$\log_{10}x=\log_{10}\frac{n\pi}{180\times60\times60}$$

$$=\log_{10}n+\bar{6}\cdot6855749.$$

When, therefore, we are working with 7-figure tables, the most convenient rules are

$$\left.\begin{aligned}\log_{10}(\sin x)&=\log_{10}n-\tfrac{1}{3}\log_{10}(\sec x)+\bar{6}\cdot6855749,\\ \log_{10}(\tan x)&=\log_{10}n+\tfrac{2}{3}\log_{10}(\sec x)+\bar{6}\cdot6855749,\end{aligned}\right\}$$

where x is equal to n seconds of arc.

For 5-figure work it is generally convenient to express the formulae in terms of minutes of arc; and, with an error not exceeding a unit in the sixth decimal place, we have

$$\left.\begin{aligned}\log_{10}(\sin x)&=\log_{10}m-\tfrac{1}{3}\log_{10}(\sec x)+\bar{4}\cdot46373,\\ \log_{10}(\tan x)&=\log_{10}m+\tfrac{2}{3}\log_{10}(\sec x)+\bar{4}\cdot46373,\end{aligned}\right\}$$

where x is m minutes of arc, and the constant $\bar{4}\cdot46373$ is found by adding $\log 60=1\cdot7781513$ to the former one.

For 4-figure work we may be content with an error not exceeding unity in the fifth decimal place; and then the formulae will be correct up to about 10° (but no change in character is needed, as we should still work with minutes).

When the constants (such as $\bar{4}\cdot46373$) are not available, we can always obtain their values by taking $\log\sin 1'$ (or $\log\sin 10''$) from the ordinary tables.* With 4-figure tables we may find that

$$\log\sin 6'=\bar{3}\cdot2419,$$

or $$\log\sin 10'=\bar{3}\cdot4637,$$

according to the subdivision adopted in the tables. In the former case we obtain the requisite constant $\bar{4}\cdot4637$ by subtracting $\log 6=0\cdot7782$; in both cases the value of $\frac{1}{3}\log\sec x$ is negligible.

4. *On the formation of J. Burgi's tables contrasted with Napier's.*

If we take as the base $c=(1+1/p)^p$, where p is large, the logarithm x of a number N will be given by

$$(1+1/p)^{px}=c^x=N.$$

* Results equivalent to these formulae are given in several sets of tables, but I do not recollect having seen the *theory* in any of the commoner text-books; and perhaps I may emphasize the fact that I have saved myself much labour (at various times) by a knowledge of the results.

Thus $$px \log\left(1+\frac{1}{p}\right)=\log N,$$

or $$px\left(\frac{1}{p}-\frac{1}{2p^2}+\frac{1}{3p^3}-\dots\right)=\log N,$$

or, on division, $$x=\log N\left(1+\frac{1}{2p}-\frac{1}{12p^2}+\frac{1}{24p^3}-\dots\right),$$

the symbols denoting natural logarithms throughout. Now in Burgi's tables, p is taken as 10^4, and, according to modern work,

$$\log 10=2{\cdot}30258509.$$

Thus Burgi's value for $\log_B 10$ should be

$$\log_B 10=2{\cdot}30258509+0{\cdot}00011513=2{\cdot}30270022,$$

which agrees with that actually tabulated by him. Accordingly the present method gives a simple means of identifying the logarithms to different bases of the present type.

Some German (and other) writers have supposed that Napier's base was given by taking $p=-10^7$. This would give (as the logarithm of 10)

$$2{\cdot}302585093-0{\cdot}000000115=2{\cdot}30258498.$$

Actually Napier's result is (in modern notation)

$$2{\cdot}302584234,$$

there having been a slip in the previous arithmetic.* But a recalculation of his tables gives the modern value

$$2{\cdot}302585093.$$

That is, the suggested base is not consistent with Napier's value for log 10 (either in his actual tables or in the corrected form). *In actual fact, Napier never used a base (in the modern sense) at all*: for a short account of his actual methods of calculation reference may be made to Art. 157 of the second edition of my book on *Infinite Series*. Doubtless what has misled hasty readers (and has produced the impression that Napier's base was that stated above) is the fact that Napier's First Table gives the values of $10^7\left(1-\frac{1}{10^7}\right)^r$, where $r=1, 2, 3, \dots, 100$; but this table is merely an auxiliary for subsequent calculations; and the value for $r=10^7$ is not worked out. T. J. I'A. B.

---

611. [Gray] expressed regret at his want of mathematical knowledge, and declared to me that he had still serious intentions of applying himself to the study of it. At the same time he lamented that in the University it was usually studied to serve the purpose of taking a degree honourably, and generally laid aside afterwards, instead of being applied to the attainment of those useful and sublime sciences to which it is the only guide and conductor.

Though I have mentioned that Mr. Gray regretted his want of mathematical knowledge, yet he would never allow that it was *necessary*, in order to form the mind to a habit of reasoning or attention. Does not Locke require as much attention as Euclid? And what cause should prevent the mind unexercised in Euclid from severe attention to Locke? or from applying the powers it possesses to any other branch of knowledge? The study of mathematics certainly requires strict attention, but does it *exclusively* produce the habit of it? and is not that habit to be acquired by application of any other sort?—*Reminiscences of Thomas Gray* (the poet, Professor of History at Cambridge, 1768-71), by Norton Nicholls. Printed in *Letters of Thomas Gray*, ed. Tovey, vol. ii. pp. 287, 291 [per Mr. F. Puryer White].

---

* The actual slip is in the course of Napier's Second Table, containing $y_r=\left(1-\frac{1}{10^7}\right)^r$; the error in y_{50} is indicated on p. 446 of my book, quoted above.

SECOND CONVERSATION ON RIGHT ANGLES.

By T. C. J. Elliott.

The object of the first conversation (*Math. Gazette*, March 1928) was to suggest the advantages of using a definition of right angle different from that of Euclid for the study of "spaces" of which the dimensions can be distinguished from each other, a very common type of space. For brevity we will refer to this study as "Dimensional Geometry," and call the right angles "dimensional right angles." The second conversation is intended to suggest that "dimensional spaces" facilitate the study of Non-Euclidean Geometry.

Inventor. To-day we will lay aside the wire gratings, and confine ourselves to drawing nets with white and red chalk on various surfaces. These nets represent two-dimensional "spaces" of which the dimensions can be distinguished from each other. From our definitions of "right angle" and "straight line," it follows as a necessary consequence that any such net is a Euclidean plane (in the sense that the sum of the angles of a triangle is two right angles) no matter what surface we draw upon. For example, the surface of this sphere is a Euclidean plane so far as it is covered by the net, and so too is this dumb-bell surface.

Student. But it seems to be impossible to draw a net so as to cover the whole surface of the sphere, at any rate not without making the two dimensions of the net closed orders instead of open ones (see *Mathematical Gazette*, March 1922). There must therefore be some absolute property to distinguish the spherical surface as a whole from the plane, a difference between them which cannot be explained away by drawing nets.

I. I do not propose to enter upon that question, but to try to avoid it by confining myself to the consideration of an actual net drawn upon only a part of the surface, for there we seem to be on safe ground. For example, if I draw a net of geodetics on the sphere, and take one mesh having two Euclidean right angles at the base and two equal uprights (a Saccheri figure), this mesh is a rectangle in the language of dimensional geometry, though, from the Euclidean standpoint, the top is less than the base, and the angles at the top greater than right angles.

S. I fail to see how the net can be called Euclidean when any two white lines or red lines meet when produced.

I. There you are introducing an idea which is foreign to the geometry of a net. It is meaningless to speak of producing lines beyond the net for then we have no guide at all as to the meaning of "straight." In a net we define straight lines to be "parallel" when they have the same direction, but we say nothing about them not meeting when produced.

S. Granting that a portion at any rate of the surface of the sphere can be regarded as a Euclidean plane, can a former plane, such as the top of this table, be regarded as, say, a Lobatchewskian surface?

I. The words "equal," "greater than," "less than," convey one meaning when we use the language of physical geometry, but a different meaning when we use the language appropriate to a net. Thus it must not be supposed that the statement that one line is greater than another is necessarily inconsistent with the statement that the two lines are equal. For we may be using "greater than" in the language of one geometry, but "equal" in the language of another. Similarly, we have seen above that two lines can be "parallel" and "not parallel" at the same time. In the language of dimensional geometry the top of the Saccheri figure on the sphere is not "less than" the base, but "equal" to it. This conversion of the words "less than" to "equal" also compels us to convert "equal" to "greater than." The logical necessity for this mode of translating the one language into the other can be illustrated by considering the case of two measuring tapes, the one divided into equal steps,

the other into constantly decreasing steps. Suppose we change the meaning of "equal," and *define* the steps of the second tape to be "equal," then we are obliged to say also that the steps of the first tape constantly *increase*. Similarly, if we consider the sequence of tops of a Saccheri figure, as we increase the distance from the base, then an observer who calls the tops "equal" on the sphere is bound to hold that they constantly *increase* on the table. Thus we not only *can* regard the top of the table as a Lobatchewskian surface, but we are logically obliged to do so, if we treat our net upon the sphere as a Euclidean plane. It will be seen that in a sense "Relative Geometry" would be a better descriptive term than "Non-Euclidean," for, if we make the table Euclidean, then the sphere is Riemannian, but if the sphere is Euclidean, then the table is Lobatchewskian. I made an unsatisfactory attempt to bring out this relationship in an elementary way in the *Mathematical Gazette*, July 1925.

S. But then if you draw your net of geodetics upon a dumb-bell surface, and call that net Euclidean, are you not equally obliged to regard the top of the table as relatively a spherical surface? If you translate "greater than" to "equal," you must also translate "equal" to "less than."

I. Certainly.

S. And, in the case of two spheres of different radii, if we call a net on the smaller sphere Euclidean, the same kind of argument as above would seem to oblige us to call the larger sphere Lobatchewskian relative to the smaller?

I. Yes, that seems correct.

S. In all this Euclid's surfaces seem to be necessary to you still, because you have to use the sphere and table to draw upon.

I. That is only for the purpose of demonstration; it is not a logical necessity, for we could imagine our plane and sphere taken in a "dimensional space" of three dimensions, what some writers call a "mathematical space." Thus the arguments above in no way hinge upon the use of the ruler or compass or set-square.

T. C. J. ELLIOTT.

Peterborough.

---

612. "How do the dear girls go on? I would have them taught geometry, which is of all sciences in the world the most entertaining; it expands the mind more to the knowledge of all things in nature, and better teaching to distinguish between truths and such things as have the appearance of being truths, yet are not, than any other. . . . How would it enlarge their minds, if they should acquire a sufficient knowledge of mathematics and astronomy to give them an idea of the beauty and wonders of the creation."—Admiral Lord Collingwood to his wife. Quoted by E. V. Lucas, *The Gentlest Art* [per Mr. R. M. Wright].

613. "As to mathematics a man must be an idiot if he was to study Greek for the sake of Archimedes, Apollonius or Diophantus. In Latin or in French you may find them all regularly translated, and parts of them embodied in the works of English mathematicians. Besides, if it were otherwise, where the notions and all the relations are so few, elementary, and determinate, and the vocabulary therefore so scanty, as in Mathematics, it could not be necessary to learn Greek, even if you were disposed to read the mathematicians in that language. I see no marvel in Halley's having translated an Arab manuscript on mathematics, with no previous knowledge of Arabic; on the contrary, it is a case (and not a very difficult case) of the art of deciphering, so much practised by Wallis and other great mathematicians contemporary with Halley."—De Quincey, *Letters to a Young Man*, III. [Per Mr. E. M. Langley.]

614. A number of gravers were found [in the floor of the vestibule of Gough's Cave, Cheddar]; also awls, geometrical points, scrapers, finely worked knives, and ribbon-like blades. . . .—From a letter on Cave Man at Cheddar, by Mr. J. A. Davies, *The Times*, Aug. 6, 1928. [Per Mr. F. Puryer White.]

MATHEMATICAL NOTES.

908. [V. 1. a.] *Note on the bracket notation for connections between variables.*

As explained in the *Mathematical Gazette*, March 1928, any two variables may be regarded as "at right angles" to each other, and variables may be regarded as the dimensions of a "space." We have a "space" when there is no connection or dependence or correspondence between the variables, but, if there is such, then we have a "connection," which is a word that includes "line," "surface," etc. If three variables, A, B, and C (such as the readings of three dials of the Dial Machine), form a "space," I represent this by ABC, and, if they form a connection, by a bracket (ABC), and the same for any number of variables. Thus the bracket (AB) stands for the *statement* that A has a one-to-one correspondence to B, or, that their connection is a "line" in the "space" AB.

This suggests that to draw a conclusion from two brackets is simply a matter of writing a fresh bracket or brackets, omitting the variables in common. To give a simple illustration, the conclusion to be drawn from (AB) and (BC) is (AC), or, as it is convenient to write, $(AB)\,(BC)=(AC)$. All cases must fall in one of three classes, for two brackets can have no variables in common, or some variables in common, or the one bracket may be entirely included in the other.

Up to a point the method seems rather promising, but difficulties begin when we take the conclusions with the original premises separately, and, so far, I have not noticed a way to avoid such difficulties. T. C. J. ELLIOTT.

909. [K¹. 6. a.] *Relative Positions of Points and Lines.*

(*a*) To find the condition that two points (x_1, y_1) and (x_2, y_2) should be on the same or on opposite sides of the line $L\equiv lx+my+n=0$.

The point on the join of the given points dividing the segment in the ratio $\mu:\lambda$ is $(\lambda x_1+\mu x_2)/(\lambda+\mu)$, $(\lambda y_1+\mu y_2)/(\lambda+\mu)$.

If this point be on the given line, $\lambda L_1+\mu L_2=0$.

Hence the points are on the same or on opposite sides of the line according as L_1 is of the same sign as or of opposite sign to L_2.

(*b*) To find the condition that the point $P\,(x', y')$ should be within the acute or obtuse angle between the lines

$$L_1\equiv l_1x+m_1y+n_1=0 \quad \text{and} \quad L_2=l_2x+m_2y+n_2=0.$$

From Fig. 1 this depends on whether the points N and Q lie on opposite or the same side of P, where $PN \perp$ to L_1 meets L_2 in Q.

The line PN is $$\frac{x-x'}{l_1}=\frac{y-y'}{m_1}=\frac{r}{+\sqrt{l_1^2+m_1^2}},$$

whence $$\frac{PN}{\sqrt{l_1^2+m_1^2}}=\frac{l_1(x-x')+m_1(y-y')}{l_1^2+m_1^2}=\frac{-L_1'}{l_1^2+m_1^2}$$

and $$\frac{PQ}{\sqrt{l_1^2+m_1^2}}=\frac{l_2(x-x')+m_2(y-y')}{l_1l_2+m_1m_2}=\frac{-L_2'}{l_1l_2+m_1m_2}.$$

Hence these distances must have opposite or the same signs, *i.e.* the point P lies within the acute or obtuse angle, according as $(l_1l_2+m_1m_2)\,L_1'L_2'$ is negative or positive.

Cor. The line $$\frac{L_1}{+\sqrt{l_1^2+m_1^2}}=+\frac{L_2}{+\sqrt{l_2^2+m_2^2}}$$

is the internal bisector of the angle between $L_1=0$ and $L_2=0$ when $l_1l_2+m_1m_2$ is negative.

(*c*) To find the conditions that the point $P\,(x', y')$ may lie within the triangle ABC (Fig. 2), whose sides are $L_r\equiv l_rx+m_ry+n_r=0$, $r=1, 2, 3$.

The geometrical conditions are that the ratios AX/XP, BY/YP, CZ/ZP should all have the same sign.

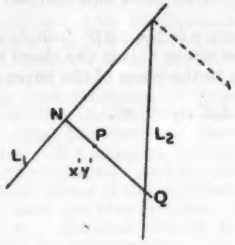


FIG. 1.

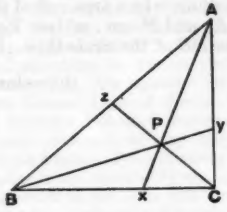


FIG. 2.

If the first of these ratios be $\mu : \lambda$ and the point A be (x_1, y_1), we have

$$(\lambda x_1 + \mu x')\, l_1 + (\lambda y_1 + \mu y')\, m_1 + (\lambda + \mu)\, n_1 = 0,$$

or

$$\lambda(l_1 x_1 + m_1 y_1 + n_1) + \mu(l_1 x' + m_1 y' + n_1) = 0.$$

Since $(x_1 y_1)$ is on $L_2 = 0$ and $L_3 = 0$,

$$\frac{x_1}{m_2 n_3 - m_3 n_2} = \frac{y_1}{n_2 l_3 - n_3 l_2} = \frac{1}{l_2 m_3 + l_3 m_2} = \frac{l_1 x_1 + m_1 y_1 + n_1}{\Delta},$$

where Δ is the determinant of the l, m, n's.

Hence

$$\frac{\lambda \Delta}{l_2 m_3 - l_3 m_2} + \mu L_1' = 0,$$

and the required conditions are that

$$(l_2 m_3 - l_3 m_2)\, L_1', \quad (l_3 m_1 - l_1 m_3)\, L_2', \quad (l_1 m_2 - l_2 m_1)\, L_3'$$

should all have the same sign.

Similarly the condition that P should lie in one of the three regions outside the triangle is that one of the above expressions should have the opposite sign to the other two.

Cor. These conditions afford tests for distinguishing analytically between the centres of the inscribed and escribed circles whose coordinates have been found from the equations:

$$\frac{L_1}{\sqrt{l_1^2 + m_1^2}} = \pm \frac{L_2}{\sqrt{l_2^2 + m_2^2}} = \pm \frac{L_3}{\sqrt{l_3^2 + m_3^2}}.$$

N. M. GIBBINS.

910. [K¹. 3. c.] *A Proof of I.* 47.

I wonder if this proof of Pythagoras' Theorem is known. It surely must be, but I have not come across it.

ACB is a triangle with a right angle at C.

Rotate the triangle through a right angle round the point C into the position A_1CB_1 (or otherwise construct the triangle A_1CB_1 by producing BC, etc.).

Then A_1B_1 is perpendicular to AB.

The area AB_1BA_1 can be looked upon as (1) the sum of two isosceles triangles of areas $\frac{1}{2}BC^2$ and $\frac{1}{2}CA^2$, or as (2) the difference between two triangles on the same base, the hypotenuse, with heights whose difference is the hypotenuse. Thus $BC^2 + CA^2 = AB^2$. The same method readily leads to the proofs of II 12 and 13.

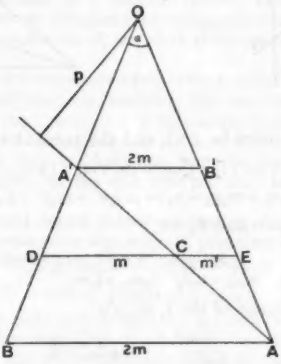
S. T. SHOVELTON.

911. . [L². 20. b.] *The Volume of a Skew Cone.*

To find the volume of the skew cone cut from a circular cone, having AA' for the major axis of its base, C the centre of its base, and OA, OA' for its least and greatest slant heights.

Its volume $=\frac{1}{3}p \times$ area πab of its base, where p is the perp^{r}. from O on AA', $a=\frac{1}{2}AA'$, and $b^2=m \cdot m'$ (see Fig.), since the minor axis is the chord through $C \perp^{r}$ to DE of the circle through DE perp^{r}. to the plane of the paper;

$$\therefore \text{ the volume} = \frac{\pi}{3} p \times \tfrac{1}{2}AA' \cdot \sqrt{m \cdot m'}.$$



Now $p \times \frac{1}{2}AA'$ is the area of the $\triangle OAA' = \frac{1}{2}OA \cdot OA \sin\alpha$, and is the mean proportional between the areas $\frac{1}{2}OA^2 \sin\alpha$, $\frac{1}{2}OA'^2 \sin\alpha$ of the triangles OAB, $OA'B'$, which areas we may call Δ, Δ';

$$\therefore \text{ the volume} = \frac{\pi}{3}\sqrt{m\Delta \cdot m'\Delta'}.$$

Now the volume of the cone of which OAB is the section

$$= \tfrac{1}{3} \text{ height} \times \pi m^2$$

$$= \frac{\pi}{3} m\Delta,$$

and similarly for the cone whose section is $OA'B'$.

$\therefore$ Finally, *the volume of the skew cone is the geometric mean between the volumes of the right cones whose slant heights are respectively the greatest and least slant heights of the given skew cone.* ALFRED LODGE.

912. . [X. 1.] *v.* Note 891, *Gazette*, vol. 14, p. 136.

Mr. Lockwood's method of approximate construction of an ellipse whose semi-axes, a, b, are CA, CB admits, I think, of simplification.

(1) His points X, Z are most easily obtained by drawing a line through (a, b) perpendicular to AB; it cuts the axes in the required points.

(2) To obtain a point Y as an intermediate centre, it is merely requisite to have $AZ+ZY=BX-XY$;

$\therefore$ $XY+ZY$ must be equal to the difference between BX and AZ.

Quite a good point for Y would be where $XY=YZ=$half that difference; but in any case Y is at the intersection of arcs of circles, centres X, Z, the sum of whose radii is the required length, viz. $BX-AZ$. ALFRED LODGE.

REVIEWS.

Theory of Probability. By W. BURNSIDE. Pp. xxx + 106. 10s. 6d. net. 1928. (Cambridge University Press.)

At the end of his life Burnside was engaged upon a treatise on Probability. He had not quite completed it when he died; his manuscript has been edited by Dr. Forsyth, whose memoir of the author, written for the Royal Society, is prefixed. The treatise, as it now appears, consists of an introduction, dealing with fundamental conceptions, and chapters on Direct Calculation, Indirect Calculation, Methods of Approximation, Probability of Causes, Geometrical Probabilities, Theory of Errors, Gauss' Law of Errors.

In so small a compass, no very profound treatment of such diverse subjects was to be expected; and in truth nothing very profound or very original is to be found. Some of the introduction is novel, and some of the indirect methods are at least not very familiar. But the reader must inevitably wonder sometimes why Burnside should have thought such a work worth publishing. Probably he began writing it in order to clear his ideas on a subject on which he first became interested in his latter days, and came to think that his solutions of some of the difficulties and obscurities that he encountered might be of interest to others approaching the subject for the first time. If this is so, the value of the book would have been very greatly increased if he had been able to complete the notes, in which he intended to draw attention to his own contributions to the theory.

But since only one of these notes appears, detailed criticism of the work would probably be misdirected. If the work is to be reviewed at any length, it can only be by making it an excuse for a general estimate of the theory that Burnside proposed to refine and polish. And it is a convenient excuse for this purpose. For Burnside appears totally unaware of most of the criticisms that have been directed recently against the whole of the orthodox mathematical theory of probability. The intention of the editor of the *Mathematical Gazette* in asking for a review of some length from one who is not a mathematician must surely have been that the opportunity should be taken of bringing these criticisms to the attention of any other mathematicians who may be similarly ignorant.

They concern mainly the definition of probability. In most text-books of algebra a definition is adopted closely following that of Poincaré:—The probability of an event is the ratio of the number of favourable cases to the number of possible cases, provided that all the cases are equally probable. Burnside substitutes for this definition the following "rule for calculating calculable probabilities":—The results of a trial or choice, or the trial itself, or both the trial and the results, are subject to such conditions that, wherever whenever and by whomever the trial is made, there are just n possible results, of which one must occur and only one can occur. If in n_A of these results the condition A is satisfied, while in the remaining $n - n_A$ it is not satisfied, the probability that the condition A is satisfied when a trial is made, is n_A/n; provided that each two of the n results are assumed to be equally likely.

This rule is not a definition in form, but it is in substance, to this extent at least. Since it is the only rule offered, we may conclude that if Burnside asserts, as the result of calculation, that some probability is n_A/n, he means that a certain trial has just n possible results, assumed to be equally likely, and that in n_A some condition A is satisfied. The verbal niceties, and the subtlety with which (in a note) a distinction is made between all events being equally likely and each two being equally likely, we may neglect; but in view of them, we may surely assume that Burnside means exactly what he says.

Let us then interpret the results of some calculations. The very first concrete example that is offered brings us to a standstill. Burnside calculates the chance of holding a suit of 5 cards or more, and finds it to be a fraction of which the denominator is $\frac{52!}{39!13!}$. He then proceeds:—"Hence the chance of holding a long suit is ·649." Now ·649 means 649/1000; according to the rule, the statement that a probability is 649/1000 means that there are just 1000 possible results of the trials; the calculation indicated that there were $\frac{52!}{39!13!}$ possible

results, or many, many millions. One statement is not a consequence of the other; it is a flat contradiction of it; the possible results cannot number both 1000 and many millions.

Burnside might possibly have admitted that here he had made a slip; but that admission will not save him. For the objection is really that his definition does not permit him to make any numerical approximation at all; if he admits that, how can he defend a chapter on Methods of Approximation, and the use of the infinitesimal calculus to which they lead? Nor can he evade the objection by following Poincaré more closely, and defining probability as a ratio rather than as a fraction. For Poincaré's definition inevitably involves that a ratio representing a probability shall always be commensurate with 1, and yet in the chapter on geometrical probabilities ratios that are not commensurable (involving π) are freely assigned to represent probabilities.

A few pages later a rather different difficulty appears. A problem is set of which this is a special case. There are two boxes of which one contains 2 white balls and 1 black, the other 3 white balls and 1 black. It is assumed to be equally likely that either of the boxes will be chosen and that either of any pair of balls in the same box will be chosen. The conclusion is that the probability of choosing a black ball is 7/24. This means, according to the rule, that there are 24 possible results of the trials and that in 7 of them a black ball is chosen. But what are these 24 results? Surely there are only 7 possible results, namely the 7 balls, and in only 2 of them is a black ball chosen. The probability need not be 2/7: for the 7 possible results may not be equally likely; but if it is not 2/7, it cannot be anything else, according to the rule.

The inconsistency in this example can be traced to a definite *non sequitur*. The general proposition from which the conclusion is deduced is

$$p_B = \Sigma p_{(A_r)B} \cdot p_{A_r}. \qquad (1)$$

Here the A_r's are conditions of which one and only one must accompany each occurrence of B, and $p_{(A_r)B}$ is the probability that B will occur when it is already known that A_r (and not any other A) has occurred. This general proposition has been proved when all the events in which B occurs are equally likely; the assumption that they are equally likely is actually emphasised in the proof. It is applied to a case when they are not all equally likely. In the deduction of the same conclusion from Poincaré's definition, the fallacy is often more skilfully concealed; but it is always there, and must be there. If the assumption that all relevant events are equally probable is part of the definition, nothing can possibly be deduced about cases in which that assumption is not made.

Let it be noted that these objections do not arise because some other definition of probability is opposed to that on which the mathematical theory is based: this definition is accepted and shown to lead to logical inconsistency. If we inquire how such obvious inconsistencies have escaped minds so acute as those of Poincaré and Burnside, the answer must surely be that it is they who have unconsciously opposed another definition to that which they have laid down formally. Their definitions do not convey what probability really means to them; at the back of their minds there is another definition; and they have been led into error by assuming that propositions true of what they really mean by probability are true of probability as they have defined it.

The real meaning of probability, to them as to everyone else, is an experimentally determined ratio between the number of times in which an event actually happens and the number of times in which it might happen. Since it is experimentally determined, numerical approximation is appropriate to it: for such approximation is an integral and necessary part of experimental measurement; if it were not permissible, numerical laws, to the formulation of which measurement is directed, could not be valid. Again, Burnside's proposition (1) is true of experimentally measured probabilities, because circumstances such that the probabilities p_B, p_{A_r}, $p_{(A_r)B}$ can be determined at all are also such that this proposition must be true; the number of times in which B happens must be the sum of the number of times in which it has happened in conjunction with each of the A's.

The question is therefore why mathematicians do not base their theory upon

the definition that they, and everyone else, mean. The answer is, of course, that no rigidly logical theory can be based on this definition. The obstacle has often been discussed in various ways, and there is no need to expound it fully here. Briefly it is that the ratio which measures probability experimentally is not the ratio in any defined finite number of trials; because the ratio is necessarily somewhat different for different numbers of trials. And it is not the ratio in an infinite number of trials (Burnside has pointed out logical objections to such a definition, even if it were otherwise satisfactory); because no experiment can extend to an infinite number of trials. Mathematicians have attempted to evade this difficulty by adopting a definition that is applicable only to a small and special class of probabilities, in which it is possible to reduce the empirical element that inheres in the meaning of probability to an assumption of equal probability. They have failed to notice that this class is small and special, and that, by confining the definition to it, they have debarred themselves, at the pain of logical inconsistency, from considering at all most of the problems that are of any practical or intellectual interest.

There is a very simple way out of the dilemma, which involves very little alteration in existing practice. Most writers use their definition only to establish a few general propositions, on which all the rest of their treatment is based. Even Burnside, though he insists upon the importance of direct calculation by simple enumeration, adopts this course; all but a very few of his calculations are actually based upon equation (1), together with

$$p_B = p_{AB} + p_{A'B} \qquad (2)$$

where A' means not-A. Why should not these few fundamental propositions be made the definition of probability "by postulate"; why should not probability be defined for mathematical purposes as that magnitude characteristic of events of which these propositions are true. Rigid consistency could then be attained; nothing of mathematical interest would be lost; the empirical element would be separated completely from the logical and concentrated in the assignment of the probabilities of the elementary events of which the more complicated are composed. Since the conclusions would hold however these probabilities were assigned, the scope of the mathematical theory would be greatly extended; some reality might be introduced into the conventional estimates of card chances by the recognition that imperfect shuffling actually makes unequal probabilities that the conventional theory is forced to treat as equal.

If this course were adopted, the choice of the fundamental propositions would become a matter of importance and interest. In this matter Burnside's work displays some originality; for his fundamental propositions (1) and (2) are not those of most authors, who usually follow Poincaré in using the propositions concerning the addition and multiplication of probabilities. The second of these propositions is true only if the probabilities are "independent"; and some difficulty is usually experienced in defining independence. Burnside clearly thinks that his procedure avoids these difficulties and enables him to define independence in terms of his fundamental propositions. If that is true, it is important. No opinion on the matter will be offered here; it is mentioned merely to indicate that, by refraining from attempts to define probability in logical terms, mathematicians will not deprive themselves of the possibility of exercising their powers of logical analysis. NORMAN CAMPBELL.

Grundzüge der Theoretischen Logik. By D. HILBERT and W. ACKERMANN. Pp. viii + 120. 7.60 m. Bound, 8.80 m. 1928. (Springer, Berlin.)

The critical work on the foundations of Analysis during last century has led to investigations in fundamental logic during the last decades. The first germs of the modern movement are found in Boole, whose work was continued by C. S. Peirce and Schröder. Partly independent of these researches, we have the various editions of Peano's *Formulario*, wherein Peano and his collaborators succeeded in putting a number of mathematical theories into logical symbolic form; this work was based on a most profound analysis of reasoning and its aim was to express all mathematics in a completely symbolic way without the use of any ordinary language. Meanwhile the disturbing

contradictions that had arisen in the theory of sets by a too naïve use of the notion of class had necessitated a closer scrutiny of logical principles, and this was carried out in Whitehead and Russell's *Principia Mathematica*, with the help of a development of Peano's symbolism.

Hilbert and his followers also use Peano's ideas, but with an aim and method different from that of previous investigators, and suggested by the axiomatic treatment of other mathematical theories. Suppose, *e.g.* that we have a set of axioms which are sufficient to characterise the real numbers; in these axioms we shall have a set of elements, whose nature is of no importance, and a set of compositions and relations between these elements (denoted by $+$, $\times$, $<$ and so on) whose nature is also of no importance, provided they obey the laws set forth in the axioms. Then the process of showing any theorem about real numbers can be regarded simply as a game, whose rules are given in the axioms; the game itself may be regarded as devoid of meaning, but the logical notions, *e.g.* "implies," "there is," and so on, used in playing the game, must be regarded as having their ordinary meaning.

Hilbert uses this method in dealing with the axioms of logic and arithmetic. These are set down as far as possible in symbolic form, the rest serve as rules for operating on these symbolic axioms. The symbols are then regarded as emptied of meaning: they are then merely pieces in a game, like the pieces in chess. We enquire as to what theorems can be deduced from these axioms, *i.e.* what combinations of symbols can be reached by following the rules and starting from the axioms, and whether the axioms lead to contradictions, *i.e.* whether certain forbidden combinations, such as $0 \neq 0$, can be reached following the rules. In considering these questions, ordinary logic is used in arguing about combinations of symbols, as it might be used in chess; the terms of ordinary logic have their usual meaning; the symbols of symbolic logic which were at first regarded as representing these terms are now regarded as emptied of meaning. Hilbert calls this method metamathematics.

The book under review is an introduction to a promised larger work dealing more fully with Hilbert's theory. It treats of the calculus of propositions and of propositional functions, the foundations of mathematics being kept in view throughout. Towards the end of the book the well-known paradoxes are thoroughly examined, the doctrine of types described, its influence on the theory of the Dedekind section shown, its imperfections pointed out, and a remedy briefly sketched. For a full treatment of the latter and its application to the foundations of mathematics we must await the larger work.

Hilbert's papers on this subject are well known, but being mostly in lecture form, they are sometimes rather sketchy, and this first instalment of a full and authoritative exposition makes us hope the sequel will soon be published. One criticism of the printing The contradictory of a proposition is denoted by a horizontal bar above the proposition. This bar is seldom thick enough to be easily seen, and is sometimes missing altogether. H. G. F.

The Decline of the West. By OSWALD SPENGLER. Authorised translation from the second German edition, with notes. By C. F. ATKINSON. Pp. xviii + 443. 21s. 1926. (Allen & Unwin.)

The scope of this book is wider than the title suggests. Spengler's object is not merely to show that "things are not what they were." He is not one of those whose sheer love of gloom leads them to depreciate the manners, morals and achievements of the present generation. He is not even solely concerned with our own times or our own civilisation. His object is nothing less than to establish a new view of world-history, more comprehensive than any put forward previously. And world-history is not, for him, primarily a matter of war and politics, nor of social, artistic, or scientific development. It is the history of human thought and desire, possibilities and actualities, or as he would say, "soul and world." All those outward forms of history are regarded as expressions of the essential "ideas" of the various cultures, and they are co-ordinated into one vast scheme, to serve Spengler's main object, which is the study of the *form* of world-history.

And here lies the appeal of the book and the reason why a German edition of 90,000 copies was justified. Every reader, be he artist, historian, politician,

mathematician, can see his own subject fitting into the great scheme. Spengler is equally at home in Mathematics, Music, Philosophy, Theology, and many other of man's forms of self-expression. That is why his book is so thoroughly convincing.

In the forefront of the vast body of facts and ideas which form the development of his main theme, he places a chapter entitled "The Meaning of Numbers." To every culture belongs its own number-idea, and on that its written "mathematic" is built up. He makes interesting references to the number-sense of Indians, Egyptians, Australian natives, but his main line is a comparison and contrast of the Western and the Classical mathematics. The Classical number-idea, first expressed by Pythagoras, was essentially confined to the positive integers. They had no zero, and irrational numbers were a mystery not to be meddled with. Their numbers were like a row of Greek statues. Their geometry, again, was essentially concerned with magnitude. A line was that which had length, that is, a finite length. A line of indefinite length meant nothing to them—it would have been a contradiction in terms.

How different the geometry of Descartes, the Western Pythagoras; the "imaginary numbers" and the "functions" of his successors! The Western mathematician is concerned not with magnitudes but with relations. In his hands geometry ceases to be metrical and is not even confined to three dimensions. It is only our extraordinary reverence for the Classical culture that has made us pretend that our mathematic has grown out of that of the Greeks. There is the same difference between our functions and their numbers as between our music and their statues, our Gothic cathedrals and their temples. Our ideas of complex numbers and four-dimensional geometry would have been repulsive to the true Greek. It was not in his nature to understand them, even supposing they had been presented to him.

Moreover, the Classical mathematic is just as alien to us as ours would have been to the Greeks. We have built ours not by developing the Classical ideas, but by destroying them, and we have done this not willingly, but from inner necessity.

* * * * * * * * * * * * *

What, then, of the mathematics we teach in school?

We have given up Euclid as such, and we have substituted a curious mixture that is neither one thing nor the other. Starting from definitions some of which mean nothing to the modern mind, we follow roughly the lines of Euclid for a certain distance, and then break off into an entirely new conception of a line as the path of a moving point. After this excursion we go back to a semi-Euclidean treatment of areas, clinging to geometrical statements of theorems which we usually think of algebraically, as mensuration formulae, and relegating theorems which were an essential part of his system into a class called "Illustrations of algebraical identities." In the treatment of circles we introduce the limit, an altogether modern idea, and in proportion we follow Legendre's advice and call in the aid of arithmetic rather than bore our pupils with Euclid's theory of magnitudes in proportion. Worse than all this, we teach Euclidean constructions, but not the logical reason for them.

Is it not time for the farce to be given up? Let Euclid be left to boys in the Classical Sixth, who should study Greek mathematics along with Greek philosophy, Greek politics and Greek art. The ordinary school course in geometry should be designed with the definite object of providing an introduction to the mathematic of our own culture. Loci, limits, and variables, instead of being admitted on sufferance, should be made prominent, the whole thing centring round graph work and leading on to the Calculus. Such facts about lengths, angles, triangles and circles as must be known for everyday purposes or for use in later work can be taught rapidly, starting from an experimental standpoint, with logical connections appearing gradually and naturally, but let us not exalt this work into a solemn ritual, a religious game to be played under strict arbitrary rules. Let us instead get on as quickly as possible to the real business of teaching the mathematic of our own Western culture. It is astonishing to find how many boys are under the impression that Geometry, if not Mathematics as a whole, has practically stood still since the time of Euclid. Why are we content that the majority should leave school

in complete ignorance of the mathematic that is theirs, stuffed instead with a hotch-potch, some of which is, and must be, dull and dead to their Western minds? Is it not a fact that the modern boy is interested in loci, graph-work, rates of change, infinity, but hopelessly bored by triangles and Euclidean constructions? Any form will vote solid for Dynamics against Statics, for the parabola of the cricket ball against that of the suspension bridge, for a cycloid against a conic section as such. And that is all as it should be and must be.

We are beginning to give our own language a chance in school. Why not our own mathematic? Latin and Greek no longer monopolise the literary side of education, but the bad side of the classical tradition still lingers in our geometry classes, and it is time for us to recognise and admit it.

The educated man of the future will have to know about mathematics, not because he is going to be an engineer or an actuary, but because the subject represents one aspect of human thought and has to be studied *in conjunction with* art, religion, philosophy. This is what many mathematicians do not realise. Mathematics is very closely connected with these things. The well-known tendency for mathematicians to be musical is neither a myth nor an accident, but a fact of fundamental significance. Modern mathematics and music are both typical expressions of what Spengler calls "the idea" of our culture, just as the finite geometry, the whole numbers, and the statues expressed that of the Greeks.

Every mathematician should read this book, for it will take him out of himself, show him what he stands for, and what is the position of Mathematics and of his own particular mathematic in the world of thought.

Felsted School. E. H. LOCKWOOD.

Secondary School Examination Statistics. By J. M. CROFTS and D. CARADOG JONES. Pp. viii + 88. 2s. 6d. 1928. (Longmans, Green & Co.)

Considering the immense importance that has so long been attached in this country to public examinations, the general ignorance of the scientific methods of conducting them is discreditable to the teaching profession; and all the more so in view of the fact that for many years Mr. D. B. Mair, as Director of the Civil Service Examinations, has given an admirable demonstration of how the thing should be done. The excuse for the prevalence, in the management even of highly important public examinations, of haphazard and indefensible methods of assessing the candidates' merits is, presumably, that there has been no exposition of correct principles at once competent and simple enough for the average examiner to understand. The publication of this excellent little book has made that excuse henceforward untenable. In substance it is an explanatory account of Dr. Crofts' administration of the Joint Matriculation and Higher Certificate Examinations of the Northern Universities, prefaced by a statistical introduction that could hardly be beaten for simplicity and lucidity. The only important fault that could be found with this part of the work is that, in dealing with the troublesome question of probable errors, the authors have neglected the advantage of pointing out that a probable error is identical with what they have already described as a quartile deviation, and that the formula connecting probable error with standard deviation, which brings in the mysterious coefficient 0·6745, is accordingly only a more exact way of stating the simple connexion between standard and quartile deviation given on p. 26.

The exposition of the problems that arise in the conduct of a public examination and the explanations given of the correct methods of dealing with them are very clear, interesting, and informative. It was well to emphasise, pp. 44-6, that in marking papers of the "essay type" (*e.g.* English, history, geography) equally conscientious and experienced examiners will often send in widely differing assessments, and that an unflinching and drastic manipulation of the marks is, accordingly, essential in the interests of justice. The authors might with advantage to their readers have gone further into this vital question, distinguishing more fully between discrepancies due to differences in "taste," which may affect seriously the relative order of merit among candidates, and differences in the subjective scales applied by examiners, which do not influence the order in which candidates would be placed in respect of a single subject,

but affect only the marks assigned to them. The ideal method of conducting an examination in which the scripts in a single subject are divided among several marking examiners would be to eliminate the first kind of discrepancy in advance by choosing like-minded people to assess the scripts, and to correct the second kind by statistical treatment. The principle to be followed in the statistical correction is that if the candidates are divided *at random* into sufficiently large groups, there cannot be any substantial difference between the performance of one group and that of any other. Hence if the assessments of one examiner result in his group's obtaining (say) a mean of 45 marks with a standard deviation of 32, while the corresponding figures for another group are 52 and 38, the headquarters administration must remove the divergency by reducing both sets of marks to a common standard so that both may have the same mean and standard deviation. The condition that the groups must be distributed between the examiners at random is essential, but constitutes, no doubt, a somewhat serious administrative difficulty. The most trustworthy way of determining whether it has been fulfilled and whether the groups are, accordingly, statistically homogeneous, is to examine the results obtained by the same groups in a subject, such as arithmetic, in which there is little or no room for differences in marking. If the marks given to the groups have, in this case, practically the same mean and standard deviation, it is legitimate to assume that the marks earned by the several groups in any other subject ought also to be statistically identical. Assuming that the marks in a single subject have thus been standardised, the marks for different subjects are brought into a line by the same principle of statistical homogeneity. The authors explain this very clearly, though they do not go into the thorny question whether, in an examination offering alternative subjects, particular subjects should be "weighted" more highly than others. There would be a justification for such weighting if it were known that certain subjects were "soft options" taken on a large scale by candidates of lower and avoided by candidates of higher ability.

Comparisons between the performances of boys and girls are unavoidable in a treatise upon examinations. Dr. Crofts' examinations bring to light no new facts, but the treatment of the data they supply exhibits the old facts clearly and shows how the significance of the difference between the work of the sexes may be estimated. The perennial interest attached to this subject will no doubt attract many readers to a most careful little book, which, it is hoped, will be widely studied by teachers and those concerned in any way with the administration of examinations. T. P. NUNN.

The Approach to Teaching. By H. WARD and F. ROSCOE. Pp. x+204. 5s. net. 1928. (Bell.)

This is a book for the beginner in teaching, and although the authors have the elementary school teacher in view it might well be read with advantage by those who have just graduated and are about to take up schoolmastering without a preliminary course of training for the work. The book presupposes that the reader intends to think about his work and endeavour actively to acquire good "craftsmanship," which, the authors point out, means more than mere technique. Hence, although many rudimentary ideas are mentioned, they are not laboured, and the book as a whole gives a fairly comprehensive view of what is involved in teaching, without offence to the reader's intellectual dignity.

There are chapters on teacher, pupil, health, discipline, general principles of teaching, stages in education, curriculum, examinations, and activity in learning. These are discussed in a practical way on broad lines and with suitable reference to modern ideas and methods. The chapter on discipline is sane and helpful to a beginner. The one on "Arithmetic and Mathematics" deals with the former subject only; a few topics such as the teaching numbers to infants, mental arithmetic, concrete work, are discussed, and then a few points of detail mainly in connection with vulgar fractions and decimals. This chapter is necessarily sketchy from the point of view of a specialist teacher, but it suitably covers enough ground to create an atmosphere and form a basis for further thought and reading. R. S. WILLIAMSON.

MATHEMATICAL ASSOCIATION.

YORKSHIRE BRANCH.

THE Autumn Meeting of the Yorkshire Branch of the Mathematical Association was held on Saturday, 20th October, in the Mathematical Department of the University of Leeds, when 52 members and friends were present and 13 new members were elected. The Chairman of the Branch, W. Peaker, B.Sc., presided: Professor W. P. Milne, M.A., B.Sc., gave a most instructive paper on "The genus of a curve"; Mr. A. F. van der Heyden, M.A., gave a most entertaining and helpful paper on "Fads and Fancies in the Teaching of Elementary Mathematics." ARTHUR B. OLDFIELD, Hon. Sec.

A PROBLEM BUREAU.

A CONSIDERABLE number of requests for the solution of mathematical questions reach individual Members of the Council in the course of the year.

It is proposed to open what, for want of a better name, we call a Problem Bureau, to assist Members of the Association when they require solutions of questions up to the standard of University Scholarship Examinations.

The scheme is tentative.

The names of those sending the questions will not be published.

The sources from which the questions are taken should be indicated.

When questions or solutions are of unusual interest or merit, a place may be found for them in the *Gazette*.

Members who desire to avail themselves of the Bureau should address their enquiries to *A. S. Gosset Tanner, Esq., M.A., Derby School, Derby*, who has secured the co-operation of experienced teachers, and has consented to act for the present as *Director of the Problem Bureau*.

---

615. The Greeks and Gravity.—Plutarch, *De Facie in Orbe Lunae*. Pharnaces having stated the doctrine of his sect (the Stoics) that the earth is in the centre of the universe and held in position by the tendency of all heavy bodies to that point, an attempt is made by his opponent to demonstrate the absurdity of this theory by showing, among other inferences which might be drawn from it, that if heavy masses of metal weighing many tons, μύδρους χιλιοταλάντους, were dropped down a hole perforating the earth, they would remain suspended at the centre; and that if a man could take up his position at the very centre of the earth his feet and head would both be uppermost. Lucretius ridicules the Stoic view at i. 1052, *De Rerum Nat.*, where Munro remarks: Had Epicurus, while retaining his conceptions of infinite space and matter and innumerable worlds and systems, seen fit to adopt this Stoical doctrine of things tending to the centre, and so to make his atoms rush from all sides of space alike towards a centre, he might have anticipated the doctrine of universal gravity. . . . But Lucretius is right in rejecting the absurd reasons which the Stoics gave for things pressing to the centre of one finite world, in the midst of infinite void, and he well expresses here . . . their inconsistency in making some things seek and others fly from the centre. As Munro observes, the Peripatetics and some others held a similar doctrine to that of the Stoics.—Bensley, *N. & Q.* X. ix. 16.

616. It is almost impossible to discredit Woodhouse's remark: "Whether I have found a logic, by the rules of which operations with imaginary quantities are conducted, is not now the question: but surely this is evident, that since they lead to right conclusions, they *must have a logic*."—De Morgan, *Trigonometry and Double Algebra*, p. 41, footnote, 1849 [per Mr. G. J. Lidstone].

THE LIBRARY.

160 Castle Hill, Reading.

The late Prof. **R. W. Genese**, a supporter of the Association for fifty years, disposed of some of his books when he left Aberystwyth and gave a considerable number to the Library shortly afterwards; by his express desire the residue has now been sent here. Members must combine earlier acknowledgments (vol. 11, p. 400; vol. 12, pp. 297, 358) with the following list to appreciate the full debt of the Library to Prof. Genese:

W. Ahrens	Mathematische Unterhaltungen und Spiele	1901
	Scherz und Ernst in der Mathematik	1904
F. Aschieri	Geometria Projettiva dello Spazio (2) - Hoepli 78	1895
W. W. R. Ball	Primer of the History of Mathematics	1895
A. H. L. S. Béchaux	Mechanical Geometry "An application to geometry of some propositions in statics."	1869
G. Bellavitis	Lezioni di Algebra	1875
L. Bolton	Introduction to the Theory of Relativity	1921
J. Bolyai	Science Absolue de l'Espace The famous essay on geometry without the axiom of parallels, preceded by F. Schmidt's pioneering biographical paper on the author and his father; translated into French, the one from Latin, the other from German, by G. J. Hoüel.	1868
E. Borel	Leçons sur les Fonctions Monogènes	1917
C. Burali-Forti	Géométrie Différentielle suivant la Méthode de Grassmann	1897
	Logica Matematica - Hoepli 168	1894
W. K. Clifford	Mathematical Papers Edited by R. Tucker, with Introduction by H. J. S. Smith. *The copy is imperfect, lacking two pages of the Index.*	1882
	Lectures and Essays (2) This popular edition omitted two essays which were in the large two-volume edition of 1879 and were reprinted in the *Mathematical Papers* also.	1886
	Common Sense of the Exact Sciences (2) Completed posthumously by K. Pearson with the aid of unfinished notes.	1886
	Seeing and Thinking (2) A course of popular lectures, published posthumously from reporters' notes.	1880
L. Couturat	L'Algèbre de la Logique (2) - Scientia 24	1914
M. Demartres	Cours d'Analyse; III: Equations différentielles et aux dérivées partielles *Has any member the earlier volumes to give?*	1896
A. Demoulin	Application d'une Méthode Vectorielle à l'Étude de divers Systèmes de Droites	1894
A. Einstein	Geometrie und Erfahrung	1921
	Grundlage der allgemeinen Relativitätstheorie	1916
	Vier Vorlesungen über Relativitätstheorie Delivered in Princeton.	1922
A. J. Ellis	Algebra identified with Geometry	1874

L. EULER	Drei Abhandlungen über Kartenprojection 1777 Ostwald 93	1898
H. FEHR	Application de la Méthode de Grassmann à la Géométrie Infinitésimale - - - - - -	1899
V. FISCHER	Vektordifferentiation und Vektorintegration - -	1904
G. FONTENÉ	Géométrie Dirigée - - - - - - - -	1897
A. FÖPPL	Technische Mechanik ; IV : Dynamik (2) - - *An odd volume. Can any member supply others ?*	1901
J. FRISCHAUF	Absolute Geometrie - - - - - - -	1876
C. F. GAUSS	*Demonstratio Nova Theorematis omnem Functionem . . . Integram . . . in Factores . . . resolvi posse* - Gauss' first proof of the fundamental theorem of algebra. A translation into German is in No. 14 of Ostwald's *Klassiker.*	1799
F. GRAEFE	Theorie der Quaternionen - - - - - -	1883
H. G. GRASSMANN	Gesammelte Mathematische und Physikalische Werke (6 vols.) - - - - - - - Edited by F. Engel, who adds a Life and a Bibliography.	1894-1911
	Die Lineale Ausdehnungslehre (2) - - - - The text of the original edition of 1844 reprinted, with three Appendices.	1878
	Die Ausdehnungslehre. Vollständig und in strenger Form bearbeitet - - - - - -	1862
R. GRASSMANN	Die Ausdehnungslehre in strenger Formelentwicklung - - - - - - - - - -	1891
	Wissenschaftslehre ; II : Formenlehre oder Mathematik - - - - - - - - - -	1872
H. HANKEL	Esquisse Historique sur la Marche du Développement de la Nouvelle Géométrie - - - - The introduction to the author's *Elemente der projectivischen Geometrie in syntetischer Behandlung*, 1875, translated into French by E. Dewulf.	1885
K. HEUN	Formeln und Lehrsätze der Allgemeinen Mechanik -	1902
D. HILBERT	Principes Fondamentaux de la Géométrie - - A translation by L. Laugel from German into French, republished from the *Ann. de l'Ecole Normale Supérieure* of the same date.	1900
H. W. L. HIME	Outlines of Quaternions - - - - - -	1894
G. J. HOÜEL	Tables Numériques - - - - - - - Includes hyperbolic and elliptic functions.	1866
E. W. HYDE	Directional Calculus - - - - - - -	1890
W. v. IGNATOWSKY	Vektoranalysis und ihre Anwendung ; I - - *Another odd volume.*	1909
E. JAHNKE	Vektorenrechnung - - - - - - - -	1905
C. J. JOLY	Manual of Quaternions - - - - - - -	1905
P. E. B. JOURDAIN	Nature of Mathematics - - - - - -	[1913]
P. KELLAND and P. G. TAIT	Introduction to Quaternions - - - - - -	1873
	THE SAME (3 : C. G. Knott) - - - - - - -	1904
A. B. KEMPE	How to Draw a Straight Line - - - -	1877

F. C. KLEIN	Famous Problems of Elementary Geometry Translated from German by W. W. Beman and D. E. Smith.	1897
G. KŒNIGS	Géométrie Réglée	1895
F. KRAFT	Abriss des geometrischen Kalküls	1893
E. LAGUERRE	Géométrie de Direction Largely reprinted from various numbers of *Nouvelles Annales de Mathématiques.*	1885
C. A. LAISANT	Quaternions	1881
E. LANDAU	Anzahl der Gitterpunkte in gewissen Bereichen	1912
P. M. H. LAURENT	Théorie élémentaire des Fonctions Elliptiques Reprinted from *Nouvelles Annales.*	1882
R. LEVEUGLE	Précis de Calcul Géométrique	1920
E. LUCAS	Récréations Mathématiques ; I-II The complete work is in four volumes.	1882, 1883
A. MCAULAY	Octonions	1898
	Utility of Quaternions in Physics	1893
A. MACFARLANE	Algebra of Logic	1879
	Bibliography of Quaternions	1904
P. MANSION	Evaluation approchée des Aires Planes	1881
R. MARCOLONGO	Meccanica Razionale (2 vols.) Hoepli 352-355	1905
J. C. MAXWELL	Matter and Motion	1876
R. MEHMKE	Anwendung der Ausdehnungslehre auf die Geometrie der Kreise in der Ebene	1880
F. MÜLLER	Vocabulaire Mathématique Français-Allemand et Allemand-Français ; I *Can any member give the German-French half of this useful reference book?*	1900
C. M. NEALE	Senior Wranglers from 1748 to 1907	1907
E. H. NEVILLE	Fourth Dimension	1921
N. NIELSEN	Theorie des Integrallogarithmus	1906
H. NOTH	Arithmetik der Lage	1882
J. ODSTRČIL	Quaternionen	1879
G. PEANO	Calcolo Geometrico	1888
	Grundzüge des geometrischen Calculs Translated from Italian into German by A. Schepp.	1891
J. PERRY	Spinning Tops	1890
E. PICARD	Développement depuis un Siècle de quelques Théories Fondamentales Three lectures delivered at Clark University.	1900
S. PINCHERLE	Algebra Complementare ; II (2) Hoepli 145 *Part I was already in the Library.*	1907
	Introduzione alla Teoria delle Funzioni Analitiche	1880
J. H. POINCARÉ	Science and Method Translated from French by F. Maitland. With Preface by B. A. W. Russell.	[1914]
L. SCHENDEL	Grundzüge der Algebra nach Grassmann'schen Prinzipien	1885

V. SCHLEGEL — Die Grassmann'sche Ausdehnungslehre - - - 1896

Raumlehre; I: Geometrie; II: Moderne Geometrie und Algebra - - - - - 1872, 1875

H. SCHUBERT — Mathematische Mussestunden (2) - - - - 1904

P. SCHURINGA — Beschouwingen over de Minimum-Loopbanen:

$$\delta\int_{s_1}^{s_2}\phi(v)\,ds=0$$ - - - - - - - - 1872

L. A. SÉDILLOT — Les Professeurs de Mathématiques et de Physique Générale au Collége de France - - - - 1869

P. SERRET — Des Méthodes en Géométrie - - - - - 1855

J. B. SHAW — Synopsis of Linear Associative Algebra - - - 1907

D. E. SMITH — History of Modern Mathematics - - - - 1896
A section of the Merriman-Woodward *Higher Mathematics*.

B. SOMMER — Die Winkelcoordinaten - - - - - - - 1848

D. M. Y. SOMMERVILLE — Non-Euclidean Geometry - - - - - - 1914

S. SPITZER — Integration linearer Differential-Gleichungen - - 1874

P. G. TAIT — Quaternions - - - - - - - - - 1867

A. TISCHNER — The Fixed Idea of Astronomical Theory - - 1885
The author cannot understand how astronomers who know that the sun is in motion can be intellectually honest in talking of it as a fixed centre when discussing planetary orbits. The unusual legend "Copyright not reserved" on his title-page is a pathetic effort to give publicity to his difficulties.

E. VIGARIÉ — Géométrie du Triangle - - - - - - - 1889
"Étude bibliographique et terminologique."

E. B. WILSON — Advanced Calculus - - - - - - (1912)

J. WOLSTENHOLME — Mathematical Problems (3) - - - - - 1891

J. C. F. ZÖLLNER — Erklärung der Universellen Gravitation - - - 1882
Includes a reprint of O. F. Mossotti's essay of 1836, *Sur les forces qui régissent la constitution intérieure des corps.*

Analyse Vectorielle Générale:

I: Transformations Linéaires - - - - - - - - 1912

II: Applications à la Mécanique et à la Physique - - - 1913

IV: Astatique - - - - - - - - - - - - 1915
The first two volumes are by C. Burali-Forti and R. Marcolongo in collaboration, the last is by M. Bottasso. All are translated from Italian into French.
Has anyone other volumes of this important series to give?

Handbuch der Mathematik (2 vols.) - - - - - - - 1879, 1881
Produced under the editorship of O. Schlömilch as a division of an *Encyklopædie der Naturwissenschaften*, and issued in 12 parts, the work was composed by F. Reidt and R. Heger.

Ueber Kartenprojection - - - - - - - - Ostwald 55 1894
J. L. de Lagrange (2), 1779; C. F. Gauss, 1825. The former's translated from French into German and the whole annotated by A. Wangerin. Gauss' paper was first *published* in 1825, but the manuscript was submitted in 1822 in answer to a prize question at Copenhagen, and this earlier date is given on the title-page.

Also text-books by J. Blaikie and W. Thomson, W. J. Dilworth (2 vols.), W. Gallatly, J. B. Lock and J. M. Child, H. A. Morgan, R. C. J. Nixon, A. Penby,

C. Taylor, H. M. Taylor, I. Todhunter, L. Zoretti; a collection of about a thousand pamphlets and offprints; *L'Intermédiaire des Mathématiciens*, vols. 1-5, 1894-1898 (a run we should be glad to extend, if any member has later volumes to present).

Journal de Mathématiques Elémentaires, vols. 17 (lacking No. 6) and 18, 1893-1894; the accession gives the Library a run of the first 18 vols. of the journal of Vuibert's (which must not be confounded with de Longchamps' journal of the same name), complete except for vol. 13 and for the number noticed above. Although the intrinsic value of the journal is small, nobody would know where to find a run if not in this Library, and any member who can repair the defects or extend the set will be doing the Association a service.

The Monist; three numbers of mathematical interest:

IX, 1 (Oct. 1898), with articles by Poincaré and Schroeder.

XII, 4 (July 1902), with articles by Mach and Poincaré.

XXVII, 1 (Jan. 1917), with articles on Grassmann (A. E. Heath) and Bolzano (D. M. Wrinch) and an article by Frege.

Rivista di Matematica, vols. 1-8, 1891-1906; Peano's valuable journal. Unfortunately the first three volumes are sadly imperfect; any reader who has odd parts of these volumes that might fill the gaps is urged to write to the Librarian.

With Prof. Genese's papers were a number of copies of his "Simple Exposition of Grassmann's Methods," offprinted from the "Gazette" of July 1927. Any reader who would like one of these is invited to write to the Librarian.

The Librarian reports also the following gifts:

From Mr. **F. J. Cock**:

L. E. Dickson — History of the Theory of Numbers; III: Quadratic and Higher Forms - - - - - - - - 1923
One-third of this volume is a report by G. H. Cresse on the number of classes of binary quadratic forms.
The Library already owed to Mr. Cock the first two volumes of this comprehensive and detailed History.

From Mr. **W. J. Greenstreet**:

B. Niewenglowski — Questions d'Arithmétique - - - - - - 1927
The elements of the theory of numbers.

From Mr. **L. Marshall**:

G. H. Hardy — Orders of Infinity (2) - - - - Cambridge 12 1924

From Prof. **E. H. Neville**:

P. M. H. Laurent — Traité d'Analyse; I - - - - - - - - 1885
We have now the first four of the seven volumes. Can anyone carry on the run?

R. S. Ball — Theory of Screws - - - - - - - - - 1876
A slim octavo, hardly to be regarded as the first draft of the *Treatise* of 1900.

From Miss **C. E. Phillips** and Miss **G. A. Platt**, collections of back numbers of the *Gazette*.

An addition to the Milne collection has been bought:

R. W. Griffin — Parabola, Ellipse, and Hyperbola, treated geometrically - - - - - - - - - - - 1879

ŒUVRES COMPLÈTES DE C. HUYGENS.

The Société Hollandaise des Sciences has filled the gaps in our set on very generous terms, and we have now the 15 volumes that have been published.

NOUVELLES ANNALES DE MATHÉMATIQUES.

A set of the first twenty volumes of this journal, 1842-1861, forming the first series complete, has been bought and bound. The Library has several later volumes in separated short runs, 1867-1874, 1876-1878, 1888-1891, 1917-1920, and other volumes would be very welcome, even if they did not actually close any of the present gaps. The journal is in some request, but unfortunately the volumes that have hitherto been wanted have always been ones that are not here.

Prof. **M. J. M. Hill** has presented a set of his papers on the theory of proportion, strongly bound into a single volume, with his Presidential Address, reprinted from the March *Gazette*, as an Introduction.

---

617. "As Prof. Vince used to say of Milton's *Paradise Lost*—very fine, but proves nothing."—Wright, *Alma Mater*, 1827, p. 221.

618. *A Curious Transcription of a Title.*—M. Newton envoya [à l'Académie des Sciences de Paris] les écrits dont nous avons déja parlé, & ceux qu'il avoit fait encore imprimer, ou qu'il publia dans la suite: comme . . . son *Analysis infinitorum à Jones, cum enumeratione curvarum tertii ordinis, & quadratura, &c*, à Londres en 1711, *in* −4°.—Moreri, *Grand Dictionnaire Historique*, s.v. Newton, Tome 7me, p. 1002, 1759. [Per Prof. E. H. Neville.]

619. "There was in particular one Tartar merchant who had been allowed to reside in Cambaluc (Pekin) during six whole months. He described that great city as lying not far from the sea coast, and for its size he said it was certainly twenty times larger than Tabriz. If so it must indeed be the greatest city in all the world, for Tabriz measures a great league and more across, and therefore this city of Cambaluc must extend to twenty leagues from one side to the other."—Clavijo, *Embassy to Tamerlane*, 1403-6 (Broadway Travellers, Routledge, 1928, p. 292).

The remarkable mathematics of this passage from Clavijo was noticed long ago by Sir Henry Yule who, quoting it in his *Cathay and the Way Thither* (2nd edition, Hakluyt Society, 1915, i. 174), adds the note "bad geometry Don Ruy!"—[Per Mr. F. Puryer White.]

620. "Qui in minimo fidelis est, et in magno fidelis est." In minimo fidelem esse magnum est. Nam sicut ratio rotunditatis, id est, ut a puncto medio omnes lineae pares in extrema ducantur, eadem est in magno disco, quae in nummulo exiguo: its ubi parva juste geruntur, non minuitur justitiae magnitudo.—St. Augustine. *De doctrina Christi*, iv. 18 [per the Bishop of Cariboo, B.C.].

621. The beginner's notion is, almost invariably, that two small quantities must be nearly equal, because they are small; and the fallacy under which they proceed is the following:—quantities which are nearly equal to the same are nearly equal to another; small quantities are all nearly equal to nothing, therefore small quantities are nearly equal to another. The mistake here lies partly in the use of *nothing* as if it were a quantity, having all the properties of quantity, partly in the supposition that quantities which differ little must be nearly equal. . . . And as to using *nothing* as a quantity in the fallacious syllogism above given, it must be remembered that, with reference to possibility of subdivision, any quantity, however small, is as distant from nothing as any other quantity, however great, is from infinity.—De Morgan, *Ratios*, P.C. 19, p. 312.

PRINTED IN GREAT BRITAIN BY ROBERT MACLEHOSE AND CO. LTD.
AT THE UNIVERSITY PRESS, GLASGOW.

BOOKS RECEIVED, CONTENTS OF JOURNALS, ETC.

December 1928.

Boswall, R. O. *The Theory of Film Lubrication.* Pp. xi+280. 12s. 6d. net. 1928. (Longmans, Green.)

Borchardt, W. T. *A First Course in Algebra.* Pp. viii+259+xlvii. 3s. 1928. (Rivington.)

Bragg, Sir W. *An Introduction to Crystal Analysis.* Pp. vii+168. 12s. net. 1928. (Bell & Sons.)

Broglie, L. de, and Brillouin, L. *Selected Papers on Wave Mechanics.* Pp. 151. 15s. net. 1928. (Blackie.)

Forsyth, C. H. *Introduction to the Mathematical Theory of Finance.* Pp. v+205. 12s. 6d. net. 1928. (Wiley: per Chapman & Hall.)

Gerlach, W., and Fuchs, F. J. *Matter, Electricity, Energy.* Pp. xii+427. 30s. net. 1928. (Chapman & Hall.)

Gunn, G. A. *The Slide Rule.* Pp. 40. 2s. net. 1928. (E. & F. N. Spon.)

Hill, T. H. Ward. *Elementary Analytical Geometry.* Pp. viii+264. 10s. 1928. (Mills & Boon.)

Jennings, C., and Tobutt, R. L. W. *Army Mathematics.* Part I. *For Army Second Class Certificate.* Part II. *For Army First Class Certificate.* Pp. ii+224. 6s. 6d. net. 1928. (Oxf. Univ. Press.)

Knopp, K. *Theory and Application of Infinite Series.* Pp. xii+571. 30s. net. 1928. (Blackie.)

Loria, G. *Matematiche greche e matematiche arabi.* (Archeion, vol. 9, Nos. 2-3, pp. 161-166. 1928.)

Nunn, T. P. *Anthropomorphism and Physics.* Annual Philosophical Lecture, Hertz Trust: British Academy. Pp. 35. 2s. net. 1928. (Milford.)

Palmer, C. I., and Miser, W. L. *College Algebra.* Pp. xiv+377. 12s. 6d. 1928. (McGraw Hill.)

Potter, F. F., and Rogers, J. W. *Common-Sense Algebra for Juniors.* Part I. Pp. viii+130. 2s. Part II. Pp. viii+131-250. 2s. 1928. (Pitman.)

Schrödinger, E. *Collected Papers on Wave Mechanics.* Pp. xiii+146. 25s. net. 1928. (Blackie.)

Schrödinger, E. *Four Lectures on Wave Mechanics.* Pp. viii+53. 5s. net. 1928. (Blackie.)

Schwerdt, H. *Einführung in die Praktische Nomographie.* Pp. vii+122. M. 3. 1928. (Salle, Berlin.)

Sierpinski, W. *Leçons sur les Nombres Transfinis.* Pp. vi+240. 40 frcs. 1928. (Gauthier-Villars.)

Turnbull, H. W. *The Theory of Determinants, Matrices, and Invariants.* Pp. xvi+338. 25s. net. 1928. (Blackie.)

Turner, H. H. *Catalogue of Earthquakes.* From Brit. Assoc. Report, 1928. Pp. 64. 2s. 1928. (Brit. Assoc., Burlington House; Univ. Observatory, Oxford.)

Westaway, F. W. W. *Geometry for Preparatory Schools.* Pp. x+202. 3s. 6d. net. 1928. (Blackie.)

Wilson, H. A. *Modern Physics.* Pp. xiv+381. 30s. net. 1928. (Blackie.)

Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universität. (Teubner, Leipzig.)

Vol. 6: Nos. 3-4, Sept. 1928.

Theorie der Eisensteinschen Reihen von mehreren Veränderlichen. Pp. 163-188. H. D. Kloosterman. *Ueber die Integration linearer, partieller Differentialgleichungen mit Konstanten Koeffizienten.* Pp. 189-197. G. Herglotz. *Topologische Fragen der Differentialgeometrie IV. Invarianten von Kurvengeweben.* Pp. 198-215. W. Blaschke and J. Dubourdieu. *Zur Isotopie Zweidimensionaler Flächen im R_4.* P. 216. E. R. Van Kampen. *Fastperiodische Folgen und Potenzreihen mit fastperiodischen Koeffizienten.* Pp. 217-234. A. Walther. *Ueber*

ein Fundamentalproblem aus der Theorie der elliptischen Modulfunktionen. Pp. 235-257. E. HECKE. *Topologische Fragen der Differentialgeometrie. Achtflachgewebe.* P. 258. J. DUBOURDIEU. *Ein logarithmenfreier Beweis des Dirichletschen Einheitensatzes.* Pp. 259-262. B. L. VAN DER WAERDEN. *Zum Phragmén-Lindelöfschen Ideenkreis.* Pp. 263-264. A. WEINSTEIN. *Neuer Beweis für die Invarianz der Dimensionszahl und des Gebietes.* Pp. 265-272. E. SPERNER. *Zum Problem von Lagrange.* Pp. 273-299. (Vier Vorträge von) J. RADON. *Ueber den Jordan-Hölderschen' Satz.* Pp. 300-302. O. SCHREIER. *Die Automorphismen der projektiven Gruppen.* Pp. 302-322. O. SCHREIER and B. L. VAN DER WAERDEN. *Drei Vorlesungen über adiabatische Invarianten.* Pp. 323-366. T. LEVI-CIVITA. *Ueber metrisch homogene Räume.* Pp. 367-376. D. VAN DANTZIG and B. L. VAN DER WAERDEN.

American Journal of Mathematics. (Johns Hopkins Press, Baltimore.)

Vol. 1 : No. 4, Oct. 1928.

Affinely Connected Function Space Manifolds. Pp. 473-517. A. D. MICHAL. *A Theorem Concerning the Affine Connection.* Pp. 518-520. T. Y. THOMAS. *Concerning Subsets of a Continuous Curve which can be Connected through the Complement of the Continuous Curve.* Pp. 521-534. W. L. AYRES. *Admissible Numbers in the Theory of Probabilities.* Pp. 535-552. A. H. COPELAND. *The Differential Invariants of Inversive Geometry.* Pp. 553-568. B. C. PATTERSON. *Expansions of the Neumann Type in Terms of Products of Bessel Functions.* Pp. 560-590. G. STEVENSON. *Canonical Forms for Ordinary Homogeneous Linear Differential Equations of the Second Order with Periodic Coefficients.* Pp. 591-612. E. SWIFT. *Periodic Orbits of Three Finite Masses about the Equilateral Triangle Points.* Pp. 615-626. H. E. BUCHANAN. *The Problem of Plane Involutions of Order $i > 2$.* Pp. 627-635. F. R. SHARPE.

American Mathematical Monthly. (Math. Assoc. of America, Menasha.)

Vol. 35 : No. 7, Aug.-Sept. 1928.

Observations on Simultaneous Quadratic Equations. Pp. 337-347. A. HENDERSON. *On the Early History of the Decimal Point.* Pp. 347-349. J. GINSBURG. *Properties of Two Points Associated with a Triangle.* Pp. 349-352. J. H. WEAVER. *A Projective Theorem on the Plane Pentagon.* Pp. 352-356. E. KASNER. *Synthetic Proof of Prof. Kasner's Pentagon Theorem.* Pp. 356-358. L. HOFMANN. *A suggestion for simplification of annuity formulas.* Pp. 358-360. W. L. HART. *A systematic method for the Solution of Simultaneous Linear Equations.* Pp. 360-363. H. G. DEMING. *A method for approximating roots of algebraic equations in pairs.* Pp. 364-367. H. G. DEMING.

Boletin Matematico. (Dr. Baidaff, Belgrano 909, Buenos Aires.)

Vol. 1 : Nos. 13-14, July 1928.

Multiplicación Abreviada. Pp. 53-54. B. I. BAIDAFF. *Comparaison des Nombres Complexes. Egalités multiples.* Pp. 59-60. B. I. BAIDAFF.

Vol. 1 : Nos. 15-16, Aug. 1928.

Leyes formales de las desigualdades complejas. Pp. 65-66. B. I. BAIDAFF. *Sobre una regla para el calculo de la vida probable.* Pp. 71-73. C. DIEULEFAIT.

Bulletin of the Calcutta Mathematical Society. (Calcutta Univ. Press.

Vol. 19; No. 3, Sept. 1928.

The Theory of Associated Figures in Hyperbolic Geometry. Pp. 101-106. R. C. BOSE. *On Plane Strain and Stress in Rotating Elliptic Cylinders and Disks.* Pp. 117-126. S. GHOSH. *On the Strong Summability (C_1) of the Fourier Series at a point where the Function has an Infinite Discontinuity of the Second Kind.* Pp. 127-134. G. PRASAD. *Note on the Difference Equations defining Enumerative Arithmetical Functions.* Pp. 135-138. E. T. BELL. *On the Solution of the General Algebraic Equation in a Symmetrical Form.* Pp. 139-142. N. GHOSH. *On the Expansion of θ in the Mean Value Theorem of the Differential Calculus.* Pp. 143-146. B. PAL. *On certain Modular Equations and Complex Multiplication Moduli.* Pp. 147-150. S. C. MITRA.

Jahresbericht der Deutschen Mathematiker-Vereinigung. (Teubner, Leipzig.)

Vol. 37 : 1-4 Heft, issued May 1928.

A. Kraser. Pp. 1-33. K. BOEHM. *Neuere Untersuchungen ueber den Fundamentalsatz in der Theorie der gewöhnlichen Differentialgleichungen.* Pp. 33-48. M. MÜLLER. *Reine und angewandte Mathematik.* Pp. 49-64. H. BEHMANN. *Ueber einen Satz von L. Fuchs.* Pp. 64-68. R. STOLZENBERG. *Math. Miszellen. XI. Ueber den Lerschen Satz.* Pp. 69-70. A. OSTROWSKI. *Geschichtliches ueber geometrische Konstructionen.* Pp. 71-74. T. KUBOTA. *Ueber ein Eliminationsproblem.* Pp. 74-82. W. F. MEYER. *Ueber schlichte Konforme Abbildungen des Einheitkreises.* Pp. 83-86. K. REINHARDT. *Ueber den Vektorenbereich eines Eikörpers.* Pp. 87-90. W. SUSS. *Ueber Striktionsgebilde.* Pp. 91-106. H. BECK. *Bermerkung zum Vortrag von H. Beck "Ueber Striktionsgebilde."* Pp. 107-112.

Vol. 37 : Nos. 6-10, Oct. 1928.

Ueber neure Untersuchungen in der Theorie der Kontinuerlichen Gruppen. Pp. 113-122. O. SCHREIER. *Die Arithmetischen Grundlagen der projektive Geometrie.* Pp. 123-148. G. SCHOLLMEYER. *Zur Geschichte der Infinitesimalrechnung bis Leibniz und Newton.* Pp. 148-187. E.

HOPPE. *Erfahrungen über die mathematische Vorbildung der Mathematikstudierenden des ersten Semesters.* Pp. 187-198. G. FEIGL. *Bericht ueber die Herausgabe des siebenten Bandes der mathematischen Werke von Karl Weierstrass: Vorlesungen über Variationsrechnung.* Pp. 199-208. R. ROTHE. *Zur Erinnerung an Karl Dochlemann.* Pp. 209-212. G. FABER. *Bemerkungen zu Grundlagenfragen.* Pp. 213-226. K. MENGER. *Ueber Vollständigkeit und Entscheidbarkeit.* Pp. 226-230. H. HARLEN. *Beweis einer Identität zwischen Binomialkoeffizienten.* Pp. 231-234. E. BESSEL-HAGEN and H. HASSE. *Ueber die Trennung der reellen Wurzeln von algebraischen Gleichungen.* Pp. 234-237. N. OBRESCHKOFF. *Neue Restgliedformen bei Funktionen mehrerer Variablen.* Pp. 237-246. W. PÜSCHEL. *Zur Schmidtschen Auflösungsformel in der Theorie der Integralgleichungen.* Pp. 246-248. G. WIARDA. *Ein Analogon zu einem Mittelwertsatze von H. A. Schwartz bei komplexen Polynomen.* Pp. 249-251. E. HOPF. *Ueber eine Erweiterung der Fourierschen Integralformel.* Pp. 251-253. L. CASPER. *Math. Misz. II. Bemerkungen zum Beweise des Budan Fourierschen und Newton-Sylvesterschen Satzes.* Pp. 254. A. OSTROWSKI. *Zur geodätischen Krümmung und Parallelverschiebung.* Pp. 257-263. E. KRUPPA. *Abbildung der linearen Linienkomplexe auf Kegelschnitte in der Ebene.* Pp. 263-274. A. FISCHER.

Journal of the Mathematical Association of Japan. (I. Mori, Tokyo Higher Normal School for Girls.)

Vol. x.: No. 2, April 1928.

L'Enseignement Mathématique. (Gauthier-Villars.)

Vol. 26: Nos. 4-5-6, April 1928.

Sur les petites oscillations d'un système possédant une fonction des forces autour d'une position d'équilibre stable. Pp. 193-199. P. APPELL. *La théorie des groupes et la géométrie.* Pp. 200-225. E. CARTAN. *Sur la représentation des groupes continus.* Pp. 226-239. H. WEYL. *Nombre dimensionnel et ensembles impropres dans le problème de Dirichlet.* Pp. 240-260. G. BOULIGAND. *Formules elliptiques pour la résolution de certaines équations de Fermat.* Pp. 260-286. E. TURRIÈRE. *Sur une formule de M. de Montessus de Ballore et les courbes binomiales.* Pp. 287-293. D. MIRIMANOFF. *Propriété d'un pinceau de normales.* Pp. 294-299. M. DUFOUR. *Logarithme d'une somme et d'une différence.* Pp. 300-302. M. PETROVITCH. *Construction du centre de courbure en un point quelconque d'une section conique.* Pp. 303-307. N. GENNIMATAS.

Vol. 27: Nos. 1-3, Aug. 1928.

Le cinquantenaire scientifique de M. Emile Picard. Pp. 5-13. A. BUHL. *Les moyennes arithmétiques dans la théorie de potentiel.* Pp. 14-26. O. D. KELLOGG. *Jacobien, divergence, formule de Green.* G. BOULIGAND. *Rotational et formule de Stockes.* A. ROUSSEL. Pp. 27-49. *Sur le moment de deux droites et son application dans la théorie des connexes.* Pp. 50-71. D. SINTSOF. *Sur la convergence des suites de fonctions quasi-analytiques.* Pp. 71-76. G. VAIJRON. *Statistiques et Probabilités.* Pp. 76-92. H. DE MONTESSUS DE BALLORE. *Une suite de nombres entiers.* Pp. 92-100. B. NIEWENGLOWSKI. *Les fonctions élémentaires et les groupes continus.* Pp. 101-105. D. POMPEIU. *Essai historique sur le parallélogramme des forces.* Pp. 106-123. A. AUBRY. *Reflexions sur l'enseignement du calcul infinitésimal.* Pp. 124-136. E. DARBESTE. *Sur un point d'intersection de six droites.* Pp. 137-141. G. FRANKE. *Sur l'addition des arcs.* Pp. 142-143. L. PELOSI.

Mathematics Teacher. (Yonkers, N.Y.)

Vol. 21: No. 4, April 1928.

Mathematics and Modern Life. Pp. 183-196. G. B. OLDS. *Finding Plane Areas by Algebra.* Pp. 197-203. J. B. REYNOLDS. *The Lesson of Dependence.* Pp. 214-218. D. E. SMITH. *Contributions of Mathematics to Modern Life.* Pp. 219-226. C. C. CAMP. *The Graph as a means of picturing Relationships.* Pp. 227-238. H. C. CHRISTOFFERSON. *The New Mathematics as a part of the New Education, its Nature and Function.* Pp 238-243. W. F. DOWNEY. *Mathematics and Sunshine.* Pp. 245-252. H. E. SLAUGHT. *Time in Relation to Mathematics.* Pp. 253-258. D. E. SMITH. *Mathematics in Modern Business.* Pp. 259-267. E. CLARKE. *Applications of Indeterminate Equations to Geometry.* Pp. 268-273. M. O. TRIPP. *Mathematics in Science.* Pp. 273-279. H. W. TYLER. *Geometry Notes.* Pp. 280-291. M. M. S. MORIARTY. *Opportunities for Transfer Making in Elementary Mathematics.* Pp. 292→.

Vol. 21: No. 6, Oct. 1928.

The Evolution of Numbers—An Historical Drama in Two Acts. Pp. 305-315. H. E. SLAUGHT. *The Influence of General Mathematics on the Subject Matter of Mathematics and on the Theory and Technique of the Teaching of Mathematics.* Pp. 316-325. L. BLANE. *The Maintenance of Fundamental Skills in the Junior High School.* Pp. 326-335. H. E. BENT. *Mathematics One Hundred Years Ago.* Pp. 336-343. H. G. MESERVE. *Mathematics and Modern Life.* Pp. 350-352. H. C. BARBER. *Is Geometry possible?* Pp. 353-356. J. F. STATHAM. *The Position of the High School Teacher of Mathematics.* Pp. 357-362. G. C. EVANS.

Memoria. (Universidad Nacional de La Plata.)

1925-1928. No. 84, April 1928.

Monatshefte für Mathematik und Physik. (Leipzig Acad., Verlagsgesellschaft, M.B.H.)

Vol. 35: No. 1, issued April 1928.

Ueber einen gemischten Tensor A_{ik}. Pp. 1-8. R. WEITZENBÖCK. *Bestimmung einer Riemannschen Metrik durch Krümmungseigenschaften.* Pp. 9-24. J. RADEN. *Zur Topologie*

berandeter Mannigfaltigkeiten. Pp. 25-44. H. TIETZE. *Ueber konvexe Kegelflächen.* Pp. 45-48. K. ZIEGLER. *Darstellung binärer Formen durch Paraboloide.* Pp. 49-82. W. SCHMID. *Ein Beweis des Hauptsatzes der Theorie der Matrizes.* Pp. 83-86. L. SCHRUTKA. *Ueber das vollständige Formensysteme der F_n im R'_m.* Pp. 87-110. W. MAYER. *On the Geometry whose Absolute is a ruled Quadric.* Pp. 111-128. J. PIERPONT. *Bestimmung der Typen derjenigen fünfgliedrigen Gruppen in beliebig vielen Veränderlichen, die nicht integrabel sind.* Pp. 129-138. H. SCHATZ. *Ueber Scharen von Osculationskurven der geodätischen Linien eines Flächenpunktes.* Pp. 139-142. J. KILGA. *Die Natur des Reduzibilitätsaxioms.* Pp. 143-146. F. WAISMANN. *Zur Geometrie der totalen Differentialgleichungen.* Pp. 147-158. H. GMEINER. *Bemerkung zu der Mitteilung von L. v. Schrutka: Ueber eine besondere Anwendung der linearen Integralgleichungen.* Pp. 159-162. L. KOSCHMIEDER. *Richtigstellung.* Pp. 153-164. L. VICTORIS.

Vol. 35 : No. 2.

Ueber die Symmetrie in den Zusammenhangzahlen Kombinatorischer Mannigfaltigkeiten. Pp. 165-174. L. VICTORIS. *Die n-gliedrigen Kontinuisierten Matrizen und ihre Anwendung auf Integral- und Integradifferentialgleichungen.* Pp. 175-196. P. HEBBONI. *Emil Müllers Leben und Wirken.* Pp. 197-218. E. KRUPPA. *Ueber einen grundleyenden Satz der Invariantentheorie.* Pp. 219-222. W. WIRTINGER. *Ueber die Geometrie von Laguerre.* Pp. 223-234. *Ueber Quasikomponenten höherer Ordnung.* Pp. 235-238. S. POINTNER. *Sur une décomposition d'ensembles.* Pp. 239-242. W. SIERPINSKI. *Elementarer Beweis des Egoroffschen Satzes.* Pp. 243-248. F. RIESZ. *Irrationale Kovarianten und elliptische Funktionen.* Pp. 249-288. E. STUDY. *On certain accessible points of plane continua.* Pp. 289-304. G. T. WHYBURN. *Einige Satze, die sich auf das Vorzeichen einer ganzen rationalen Funktion beziehen; nebst Anwendungen dieser Satz auf die Abschnitte und Abschnittemittelwerte von ebenen und räumlichen harmonischen Entwickelungen und von beschränkten Potenzreihen.* Pp. 305-344. L. FEJÉR.

Nieuw Archief voor Wiskunde. Nieuwe Opgaven. (P. Noordhoff, Uitgever, Groningen.)

Ser. 2. Vol. 15 : No. 4, issued May 1928.

Twee eenvoudige stellingen over regeloppervlakken. Pp. 289-292. W. VAN DER WOUDE. *On the solution of Mathieu's equation.* Pp. 293-301. H. BREMEKAMP. *Neue Begrundung der Eliminations- und Resultantentheorie.* Pp. 302-321. B. L. VAN DER WAERDEN. *Ueber das Riemannsche Integral.* Pp. 321-330. J. RIDDER. *The converse of Fermat's Theorem.* Pp. 330-334. N. G. W. H. BEEGER. *On a large prime number.* Pp. 334-336. N. G. W. H. BEEGER. *Sur la décomposition des nombres de la forme $y^n \pm 1$.* Pp. 336-346. O. BOTTEMA. *Over de beweging van een onveranderlijk vlak in zich zelf met twee graden van vrijheid.* Pp. 346-376. H. J. E. BETH. *Une propriété des fractions continues.* Pp. 377-378. J. WOLFF. *Over reeksen van rationale functies.* Pp. 379-382. J. WOLFF. *Ueber zweie Integrale.* Pp. 383-385. L. GEUNS.

Nieuwe Opgaven.

No. 14 : Nos. 162-200.

Periodico di Matematiche. (Zanichelli, Bologna.)

Ser. 4. Vol. 8 : No. 3, May 1928.

Diagramma rappresentativo degli stati quantici e della formazione degli elementi nei sistema periodico. Pp. 165-174. O. M. CORBINO. *Il postulato di Campano e i fondamenti dell' aritmetica.* Pp. 175-184. M. T. ZAPELLONI. *L'uguaglianza dei triedri in Euclide.* Pp. 185-188. A. AGOSTINI. *Una dimostrazione diretta del teorema di Zenodoro.* Pp. 189-191. C. SCHOR. *Osservazione sulle serie con applicazione al calculo di π e di e^{-1}.* Pp. 192-198. P. CATTANEO. *Sulla riserva matematica dei premi puri nelle assicurazioni sulla vita.* Pp. 199-204. G. BISCONCINI.

Ser. iv. Vol. 8 : No. 4, July 1928.

Vue générale de la Physique moderne. Pp. 229-238. A. LORENTZ. *Appunti storici sul paradosso di d'Alembert.* Pp. 239-256. A. MASOTTI. *I poliedri regolari stellati.* Pp. 257-270. A. MARONI. *Calcolo di π.* Pp. 271-293. U. CASSINA.

Proceedings of the Edinburgh Mathematical Society. (Bell & Sons.

Ser. 2. Vol. 1 : No. 3, July 1928.

Note on Numerical Integration. Pp. 139-148. C. E. WOLFF. *Infinitesimal Analysis of an arc in n-space.* Pp. 149-159. R. N. SEN. *Some Properties of a Family of Curves on a Surface.* Pp. 160-165. C. E. WEATHERBURN. *Note on the Elementary Divisors of some Related Determinants.* Pp. 166-168. A. C. AITKEN. *The "Fourier" Theory of the Cardinal Function.* Pp. 169-176. J. M. WHITTAKER. *A Generalization of Circulants.* Pp. 177-181. E. T. BELL. *Apolar Triads associated with the Nodal Cubic.* Pp. 182-188. W. SADDLER.

NOTICE.

The *contents* of the Journals, etc. will no longer be published, a very few members found them useful.

NEW CAMBRIDGE BOOKS

MATHEMATICAL AND PHYSICAL PAPERS. By SIR JOSEPH LARMOR, Sc.D., F.R.S. In 2 volumes. Royal 8vo. £6 6s net.

About half of this collection of papers is of electrical character, the other half being mainly General Dynamics and Thermodynamics, including the dynamical history of the Earth, Formal Optics, and Geometry.

THE ELEMENTARY DIFFERENTIAL GEOMETRY OF PLANE CURVES. By R. H. FOWLER, M.A. **Second Edition.** Demy 8vo. 6s net. (Cambridge Tracts in Mathematics, No. 20.)

THE EARTH. Its Origin, History, and Physical Constitution. By HAROLD JEFFREYS, M.A., D.Sc. **Second Edition.** With 3 plates and 16 text figures. Royal 8vo. 20s net.

In this new edition, seismology, which occupied a somewhat subordinate place, has been much expanded; Eddington's revised theory of stellar constitution has led to some quantitative changes in the account of the early history of the planets; recent developments in seismology have made it possible to go into much more detail about the structure of the earth. It has also become possible to be much more definite than before about the nature and degree of isostatic compensation. More attention has been given to the bodily tide, and the account of the earth's thermal history has been recast.

TRIGONOMETRY. By A. W. SIDDONS, M.A., and R. T. HUGHES, M.A. Part I, Numerical Trigonometry, 1s 9d. Part II, Algebraical Trigonometry, 2s 6d. Part III, Complex Numbers and Finite Series, 1s 9d. Part IV, Infinite Series, Products, etc., 3s 6d. Parts I and II, in one volume, 3s 6d. Parts I-III, in one volume, 4s 6d. Parts III-IV, in one volume, 4s 6d. Parts I-IV, complete in one volume, 7s 6d. **A key** to the book, entitled, *Some Hints and Solutions*, will be published shortly.

CAMBRIDGE UNIVERSITY PRESS
Fetter Lane, London, E.C. 4

THE MATHEMATICAL ASSOCIATION.

(An Association of Teachers and Students of Elementary Mathematics.)

'I hold every man a debtor to his profession: from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves, by way of amends, to be a help and an ornament thereunto."—BACON (Preface, *Maxims of Law*).

President:

W. F. SHEPPARD, Sc.D., LL.M.

Vice-Presidents:

Prof. A. R. FORSYTH, Sc.D., LL.D., F.R.S.
Prof. G. H. HARDY, M.A., F.R.S.
Sir T. L. HEATH, K.C.B., K.C.V.O., D.Sc., F.R.S.
Prof. E. W. HOBSON, Sc.D., F.R.S.
A. LODGE, M.A.
Prof. T. P. NUNN, M.A., D.Sc.
A. W. SIDDONS, M.A.
Prof. H. H. TURNER, D.Sc., D.C.L., F.R.S.
Prof. A. N. WHITEHEAD, M.A., Sc.D., F.R.S.
Prof. E. T. WHITTAKER, M.A., Sc.D., F.R.S.
Rev. Canon J. M. WILSON, D.D.

Hon. Treasurer:

F. W. HILL, 107 Enys Road, Eastbourne, Sussex.

Hon. Secretaries:

C. PENDLEBURY, M.A., 39 Burlington Road, Chiswick, London, W. 4.
Miss M. PUNNETT, B.A., The London Day Training College, Southampton Row, London, W.C. 1.

Hon. Secretary of the General Teaching Committee:

ALAN ROBSON, M.A., The College, Marlborough, Wilts.

Editor of *The Mathematical Gazette*:

W. J. GREENSTREET, M.A., The Woodlands, Burghfield Common, Reading, Berks.

Hon. Librarian:

Prof. E. H. NEVILLE, M.A., B.Sc., 160 Castle Hill, Reading.

Other Members of the Council:

W. C. FLETCHER, M.A.
Miss M. J. GRIFFITH.
F. G. HALL, M.A.
W. HOPE-JONES, B.A.
Prof. G. B. JEFFERY, M.A., F.R.S.
H. K. MARSDEN, M.A.
Prof. W. P. MILNE, M.A., D.Sc.
Prof. H. T. H. PIAGGIO, M.A., D.Sc.
Miss D. R. SMITH.
Miss L. M. SWAIN.
C. O. TUCKEY, M.A.
Miss E. WISE, M.A.

Hon. Secretary of the Examinations Sub-Committee:

W. J. DOBBS, M.A., 12 Colinette Rd., Putney, S.W. 15.

THE MATHEMATICAL ASSOCIATION, which was founded in 1871, as the *Association for the Improvement of Geometrical Teaching*, aims not only at the promotion of its original object, but at bringing within its purview all branches of elementary mathematics.

Its purpose is to form a strong combination of all persons who are interested in promoting good methods of teaching mathematics. The Association has already been largely successful in this direction. It has become a recognised authority in its own department, and is continuing to exert an important influence on methods of examination.

The Annual Meeting of the Association is held in January. Other Meetings are held when desired. At these Meetings papers on elementary mathematics are read and discussed.

Branches of the Association have been formed in London, Bangor, Yorkshire, Bristol, Manchester, Cardiff, the Midlands (Birmingham), the North-Eastern District (Newcastle-upon-Tyne), New South Wales (Sydney), Queensland (Brisbane), and Victoria (Melbourne). Further information concerning these branches can be obtained from the Honorary Secretaries of the Association.

"The Mathematical Gazette" (published by Messrs. G. BELL & SONS, LTD.) is the organ of the Association. It is issued at least six times a year. The price per copy (to non-members) is usually 2s. 6d. each. The *Gazette* contains—

(1) ARTICLES, mainly on subjects within the scope of elementary mathematics;

(2) NOTES, generally with reference to shorter and more elegant methods than those in current text-books;

(3) REVIEWS, written when possible by men of eminence in the subject of which they treat. They deal with the more important English and Foreign publications, and their aim is to dwell on the general development of the subject, as well as upon the part played therein by the book under notice;

(4) QUERIES AND ANSWERS, on mathematical topics of a general character.